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FREE OSCILLATIONS OF THIN-WALLED OPEN SECTION CIRCULAR RINGS

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OPEN SECTION CIRCULAR RINGS

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This thesis is dedicated to

Bonnie

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

Figure | Page

LIST OF PRINCIPAL SYMBOLS

Latin

A	Cross-sectional area of the ring
A_1, A_2	Lamé parameters
a_x, a_y	Coordinates of rigid-cross-section displacements origin
b	Eccentricity between the shear center and the center of gravity
b_{ij}	Linear operators in the out-of-plane matrix equation of motion
D_i	$\frac{\partial^i}{\partial s^i}\left\{\quad\right\}$, i = 1, 2, 3, or 4
D_1	Timoshenko warping constant (see Appendix A). The distinction with the D_i is clear. $D_1 = EC_w$
E	Modulus of elasticity
e	Distance from the geometric center of the cross-section to (a_x, a_y)
F	Change in energy per circumferential inch as defined in (3-59)
G	Shear modulus of elasticity
h	Thickness of the tubular cross-section
$h(\phi)$	Component of distance from (a_x, a_y) to point on the surface of the cross-section parallel to the lateral shell displacement, $w(\theta, \phi, t)$ (see Figure 4)
I_x	Moment of inertia of the cross-section about the x-axis (see Figure 1)
I_y	Moment of inertia through the center of area parallel to the y-axis (see Figure 1)

Latin Symbols

I_T	Torsional moment of inertia
I_i, $i=1,\ldots,44$	Defined in Appendix B
I_i, $i=45,\ldots,63$	Defined in (3-66)
J, I_p	Polar moment of inertia
K_{ij}	Algebraic expressions in the characteristic equation for out-of-plane oscillations
$L(\ldots)$	Linear stiffness operator
$M_x,M_s,M_z(s,t)$	Moments about the radial, circumferential, and transverse axis, respectively
$M(\ldots)$	Linear mass operator
M_i, $i=1,\ldots,12$	Defined in Appendix B
M_i, $i=13,\ldots,18$	Defined in (3-66)
n	Integer indicating eigenvalue or eigenmode
$n(\phi)$	Component of distance from point on the surface of the cross-section to (a_x,a_y) parallel to the cross-sectional surface displacement
$Q_x,N,Q_z(s,t)$	Shears in the radial, circumferential, and transverse directions, respectively
$Q_m(t)$	Generalized force
R	Radius of the ring from the center line to the geometric center of the circular cross-section (see Figure 1)
s	Circumferential coordinate in units of length measured along the shear center
t	Line variable
T	Kinetic energy per circumferential length
$u,v,w(s,t)$	Displacement components in the radial, circumferential, and transverse directions, respectively

Latin Symbols

$U_n, V_n(s)$	nth spatial in-plane modes
V	Strain energy per circumferential length
$w_x, w_s, w_z(s,t)$	Loading in the radial, circumferential, and transverse directions, respectively
$W_n, \psi_n(s)$	nth spatial out-of-plane modes
$(\bar{\ })$	Indicates a constant

Greek

δ_{mn}	Kronecker delta
$\delta(\ )$	First variation of ( )
$\epsilon_1, \epsilon_2, w$	Mid-surface strains (see (3-3))
δ_ϕ	Circumferential mid-surface strains
$v(\theta,t)$	Circumferential displacement in the shell-to-ring theory (see (3-25))
θ	Spatial coordinate in the circumferential direction
θ_o	Arbitrary angle
κ_1, κ_2, τ	Changes in curvature (see (3-4))
ν	Poisson's ratio
$\xi, \eta, \psi(\theta,t)$	Deflections of the rigid cross-section
ρ	Mass density
τ	Curvature twist term
ϕ	Spatial coordinate in the plane of the cross-section (see Figure 1)
χ_x, χ_z	Curvature changes in the radial and circumferential directions
$\psi(s,t)$	Rotation of the ring about the shear center

Greek Symbols

ω_o	Breathing frequency
ω_n	In-plane and out-of-plane eigenfrequencies. The distinction is clear from the context.
$(\dot{\ })$	$\frac{\partial}{\partial t}(\ )$
$(\ )'$	$\frac{\partial}{\partial \theta}(\ )$

SUMMARY

A theoretical study of the eigenfrequencies for a complete thin-walled open-section circular ring with one plane of symmetry in the plane of the ring is conducted. The analysis is divided into two parts: a conventional theory incorporating St. Venant torsion, Timoshenko warping, torsional inertia, and the shear center eccentricity from the center of gravity for out-of-plane free oscillations and extensionality for in-plane free oscillations; a higher order theory of free oscillations derived from Novozhilov's thin shell theory by restricting the toroidal shell's cross-section so that it deforms as a rigid body in the plane of the cross-section but is free to warp out of that same plane.

The complete ring eigenfrequencies corresponding to eigenmodes of four or more nodes resulting from the conventional theory are found to differ widely from the values available in the literature even for slender rings, and this is attributed to the incorporation of the shear center eccentricity--an effect that has been previously ignored.

The reduction of the shell theory to a ring theory (called the shell-to-ring theory) is accomplished through a lengthy integration of the cross-sectional surface variable. The procedure provides a systematic incorporation of the shear effects and rotary inertia terms into the equations of motion. The disadvantages of this method are that the problem is restricted to a given cross-section (a

slit-tubular ring), and the elimination of one of the shell's spatial variables is accomplished through an inextensionality assumption that results in slightly higher eigenfrequencies than the conventional solution if the shear effects and rotary inertia terms are ignored.

Comparisons of the conventional theory and the shell-to-ring theory for out-of-plane bending predominant eigenmodes show good agreement for the first few eigenfrequencies for rings with radii ratios as large as one to five, but then the shear effects and rotary inertia terms become more pronounced and a significant difference between the eigenfrequencies develops. Comparisons of the theories for out-of-plane torsion predominant eigenmodes indicate a more pronounced difference over the range of eigenmodes considered.

Comparisons of the conventional theory and the shell-to-ring theory for in-plane vibrations show good agreement for the eigenfrequencies of specimens with small radii ratios (1/50) but demonstrate a much more pronounced difference for specimens with larger radii ratios (1/5).

It is concluded that the conventional theory incorporating the shear center eccentricity is adequate to describe the first few eigenfrequencies of a complete monosymmetric ring with a radii ratio in the neighborhood of 1 to 50. Extrapolation of this theory to include rings of larger radii ratios should be regarded as erroneous.

CHAPTER I

INTRODUCTION

The current generation of spacecraft and aircraft use longitudinal stringers and rings or curved beam segments with a wide variety of cross-sectional areas to provide skeletons for fuselages, fuel or oxidizer tanks, space capsules, and similar structures. Satellites have collapsible antennae composed of thin-walled open cross-section curved beams radiating from a central point and connected with webbing; various control mechanisms have moving parts consisting, among other things, of ring segments. Quite often the design of these rings and stringers must be a compromise between engineering and manufacturing; consequently these members frequently have thin-walled open cross-sectional areas. In order for aerospace vehicles to be economically feasible or carry sufficient payloads it is generally necessary to execute advanced weight saving studies. These studies usually are accompanied by or necessitate refined vibration analyses.

Rings or curved beams are a case in point. Because or the complexity of these various systems or portions of these systems, it is generally necessary to restrict the part that these members play in the overall analysis to an elementary theory. One example of this type treatment is found in Cohen [1]* and numerous other examples can be

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*Numbers in brackets refer to references listed in Literature Cited.

found in development work on systems as described above. It is the purpose of this research to isolate the ring or curved beam as such and to study the restrictions of an elementary theory in the light of a higher order theory that includes extensionality, rotary inertia, and shear effects. It is hoped that practicing engineers will be able to make better predictions as to the dynamic characteristics of rings or curved beams through a more thorough understanding of the limitations of an elementary theory.

A monosymmetric ring, i.e. a ring whose cross-sectional area contains one plane of symmetry, splits into two separate analytical problems--in-plane vibrations and out-of-plane vibrations (in-plane refers to the plane of the ring while out-of-plane refers to displacements out of the plane or transverse to the plane of the ring).

The in-plane free vibration of the complete ring neglecting rotary inertia and shear effects was first examined by R. Hoppe [2], 1871, as cited in A. E. H. Love [3, p. 452], 1892. The frequency for a complete ring vibrating with n wave-lengths, n being any integer greater than unity, is given by the equation

$$\omega_n^2 = \frac{EI_y}{\rho A R^4} \frac{n^2(n^2-1)^2}{(n^2+1)} \qquad (1\text{-}1)$$

where the equation has been modified for various solid, symmetric cross-sections and conforms to the sign convention described in Figure 1. Lamb [4], 1888, determined the symmetric and asymmetrical natural frequencies for a shallow "free-free" incomplete ring. He assumed that

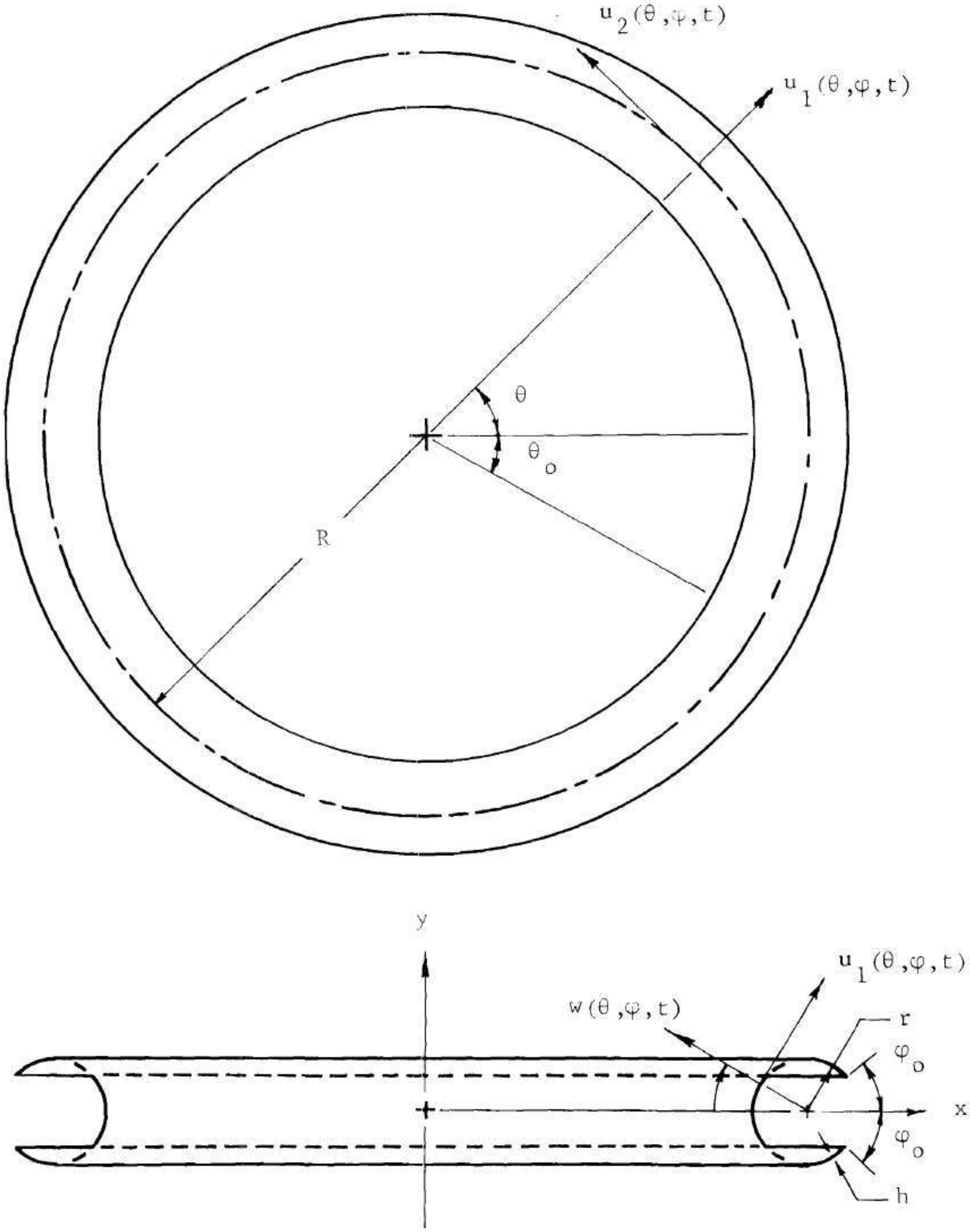

Figure 1. Description of the Slit-Tubular Ring: Its Spatial Parameters and Shell Displacements.

the extension of the neutral axis of the ring is zero, and theories incorporating this assumption are hereafter referred to as inextensional theories. Den Hartog [5], 1928, in an effort to reduce the noise of electrical machinery, analyzed vibrations of inextensional incomplete rings in the plane of the curvature subject to "clamped-clamped" or simply supported boundary conditions. He obtained eigenfrequencies to within 10 per cent of experimental values. In the same year he published a frequency study [6], 1928, for an inextensional arc subjected to hinged and clamped boundary conditions using a Rayleigh [7], 1894, solution. Den Hartog's theory postulates a shallow arc, that is one in which the circular ring segment encompasses less than 60°. The "clamped-clamped" incomplete ring was analyzed by Waltking [8], 1934, where he attempted to study the effect of the inextensionality assumption. A summary of the results of Den Hartog and Waltking may be found in a monograph by Federhofer [9], 1950. Reissner [10], 1956, examined the problem of inextensionality for a "slightly" curved bar and concluded that for shallow arches, the effect of longitudinal inertia is negligible. In 1957 solutions based on the inextensionality restriction were investigated more closely by Philipson [11]. He concluded that rotary inertia is of the same order of magnitude as extensionality and went on to conclude that the nonextension assumption is valid when

$$pq \ll 1$$

where

$$p = \frac{I_y}{AR^2} , \qquad q = \frac{\rho R^4 \omega_n^2}{EI_y} . \tag{1-2}$$

This relation is certainly satisfied for a thin ring and sufficiently small eigenfrequencies. Philipson further concluded that extensionality is important in forced vibration if the problem involves a constant applied force. The free-free ring was extended to a non-shallow geometry by Morley [12], 1958, in computing the first ten eigenvalues of the symmetric and asymmetric modes. Archer [13], 1960, examined the basic equations of motion as given by Love [3] with the addition of terms representing damping. He also examined the problem of an incomplete ring clamped at one end and given a prescribed, time-dependent displacement at the other end. A Rayleigh-Ritz solution in conjunction with Lagrangian multipliers was used by Nelson [14], 1962, to obtain natural frequencies for symmetric and asymmetric modes for both extensional and inextensional vibration. The results agree with previously referenced simplified solutions. A very complete study of the inextensional and extensional deformation theories for the in-plane vibration of thin circular rings is given by Lang [15], 1962, and [16], 1963. Additional studies for non-uniform thin circular rings are found in [17] and [18], 1962, by Lang and Reed. The writer is unaware of any reference that examines thin-walled complete or incomplete in-plane ring vibrations with the inclusion of shear effects and rotary inertia with or without the inextensional assumption.

The problem of the out-of-plane free vibration of the incomplete circular ring was first discussed by St. Venant in 1843 as described by Love [4, p. 450], 1892. The theory was more fully developed by Mitchell [19] in 1890. His frequency equation is

$$\omega_n^2 = \frac{EI_x}{\rho AR^4} \frac{n^2(n^2-1)}{n^2 + \frac{EI_x}{GJ}} \tag{1-3}$$

where a contemporary nomenclature has been used. Peterson in [20], 1930, analyzed out-of-plane vibration of circular gears and came up with an empirical formula for the natural frequency of a gear in terms of its dimensions and material. His paper explains how the natural frequencies can be changed and how large damping can be built into the system. Transverse vibration of incomplete rings is further explained by Brown in [21], 1934, in a paper analogous to Den Hartog's in-plane study of electric motor vibration. Volterra in [22], 1955, analyzed the equations of motion for incomplete rings taking into account the influence of shear and of rotary inertia by assuming that during motion the sections originally normal to the axis of the bar remain plane. This assumption means that Volterra has neglected warping, a phenomenon that has been shown to be of more significance than rotary inertia and shear effects if the ring is a thin-walled specimen. Volterra, of course, intended for his paper to be confined to rings of solid cross-section. A theory for bending and torsion of a plane beam whose axis forms a circular arc of small curvature is derived by Vlasov [23], 1959.

Vlasov's work, however, is for buckling and not free or forced vibration. Ojalvo, in [24], 1962, examined vibrations of incomplete elastic rings under various boundary conditions and time dependent distributed loads. He gave additional information on in-plane and out-of-plane natural frequencies of "clamped-clamped" and "clamped-free" ring segments in [25] and [26], 1964. An extension of Nelson's in-plane paper [14] to out-of-plane free vibration of an incomplete circular ring is given in [27], 1963. Nelson used a Rayleigh-Ritz solution with trigonometric coordinate functions and Lagrangian multipliers to enforce clamped boundary conditions. All of the authors up to this point have restricted their cross-sections to areas in which the shear center and center of gravity coincide. Krahula, in [28], 1965, examined the bending of a uniform circular ring on an elastic transverse and rotational foundation. His deflection equations are based on change of curvature expressions taken from Vlasov [23, pp. 448-452] without reference. Krahula, in [29], went on to examine the free vibrations of a uniform ring. He derived an equation that includes warping but must be restricted to rings whose shear center coincides with the center of gravity. His frequency equation is

$$\omega_n^2 = \frac{n^2(n^2-1)^2\left(1 + \dfrac{D_1 n^2}{GI_T R^2}\right)}{\dfrac{R^4 \rho A}{EI_x}\left(n^2 + \dfrac{EI_x}{GI_T} + \dfrac{D_1 n^4}{GI_T R^2}\right)} \tag{1-4}$$

Callahan examined the natural frequencies for an elliptical ring in [29], 1965, and for a circular ring in [30], 1966. He used Mindlin's and Reissner's theory on plates in such a manner as to solve the ring problem. Callahan by the nature of his theory is restricted to rectangular cross-sectional areas and consequently to rings whose shear center coincides with the center of gravity. Callahan's work has been criticized in the *Applied Mechanics Reviews* for his comments on the convergence of the solution (no numerical work is offered) and for the amount of calculations required in a solution. The author knows of no reference in which the out-of-plane natural frequencies are determined for a thin-walled open cross-section ring whose shear center does not coincide with the center of gravity.

It is the purpose of this research to determine the linear, partial differential equations of motion with rotary inertia and shear effects for a thin-walled, open, complete, circular ring with one plane of symmetry in the plane of the ring as given in Figure 1, to solve for the eigenfrequencies, and to conduct comparative studies with elementary theories.

CHAPTER II

COUPLED VIBRATIONS OF THIN-WALLED CIRCULAR RINGS OF OPEN CROSS-SECTION

Introduction

Free vibrations of a ring can be broken into several cases depending on the geometric properties of the cross-section, i.e. whether or not the cross-section is nonsymmetric, monosymmetric with respect to the plane of the ring, or doubly symmetric with respect to the plane of the ring.

In the general situation of a nonsymmetric ring or in the cases where planes of symmetry exist at an angle with respect to the plane of the ring, the free vibration is composed of lateral and in-plane bending vibrations as well as axial vibrations coupled with torsional vibrations. By lateral it is meant vibrations perpendicular to the plane of the ring; and by axial it is meant circumferential vibrations of the ring. This situation is called "quadruple coupling" and consists of both dynamic and static coupling.

Given a monosymmetric cross-section whose plane of symmetry coincides with the plane of the ring, the free vibration splits into two cases: out-of-plane or lateral vibration and in-plane vibration. The lateral free vibration is composed of out-of-plane bending vibration statically and dynamically coupled with torsional vibration.

The in-plane vibration is composed of in-plane bending statically coupled with axial vibrations. These situations are called lateral and in-plane "double coupling," respectively.

The final case is the ring consisting of a doubly symmetric cross-section. Here as in the monosymmetric case, the free vibration splits into two cases: lateral vibration and in-plane vibration. The lateral free vibration is composed of out-of-plane bending vibration statically coupled with torsional vibration. The in-plane vibration has the same characteristics as the monosymmetric in-plane vibration, i.e. in-plane bending statically coupled with axial vibrations. There are also "double coupling" cases.

The mathematical model illustrated in this analysis is a thin-walled, open circular ring with a cross-section symmetric about a line coincident with the plane of the ring. In other words,the equations of motion for the separate problems of lateral double-coupling and in-plane double-coupling are to be derived and solved.

The concept of the shear center loses its physical interpretation in the case of the ring, but there are advantages in maintaining the mathematical definition. By taking the limit as the radius of the ring goes to infinity the well-known beam equations for coupled free vibrations are readily obtainable, and the shell derivation given in the next chapter is greatly simplified if the beam type deformations are supposed to act through the shear center. Henceforth the term shear center will mean the point with respect to the cross-section coinciding with the mathematical definition of the shear center of a straight beam with the same cross-section.

Doubly Coupled In-Plane and Out-of-Plane Equations of Motion

The sign convention, geometric parameters, and deformations of the ring are illustrated in Figure 2 where the monosymmetric cross-section selected for illustration is the same as that analyzed in Chapter III.

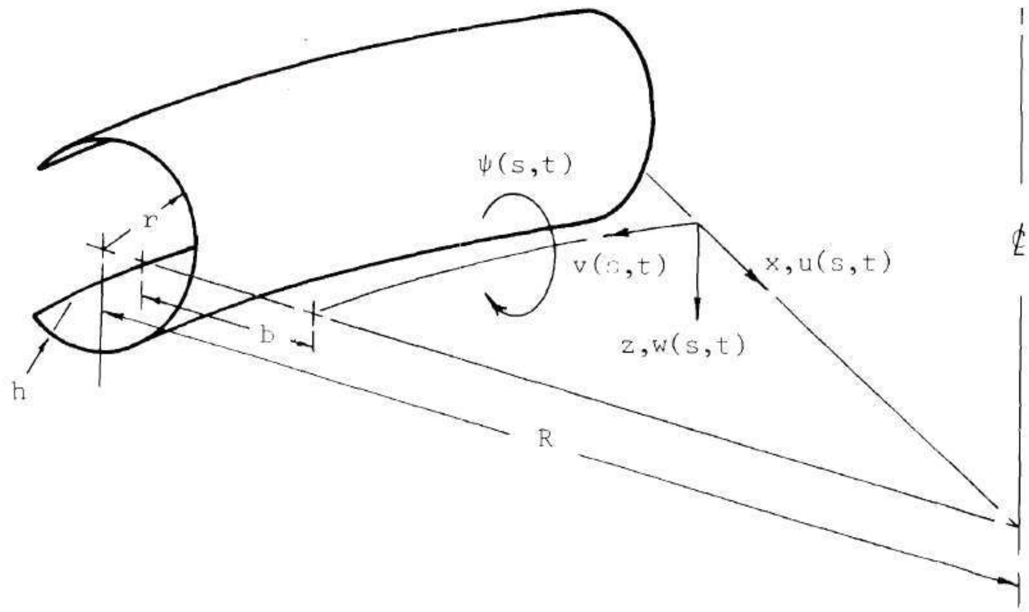

Figure 2. Ring Deformations, Geometric Parameters, and Position Variables

The equations of motion are derived from the equations of equilibrium and d'Alembert's principle. The force equilibrium equations are readily obtainable from the elemental force and moment diagrams given in Figure 3 and are

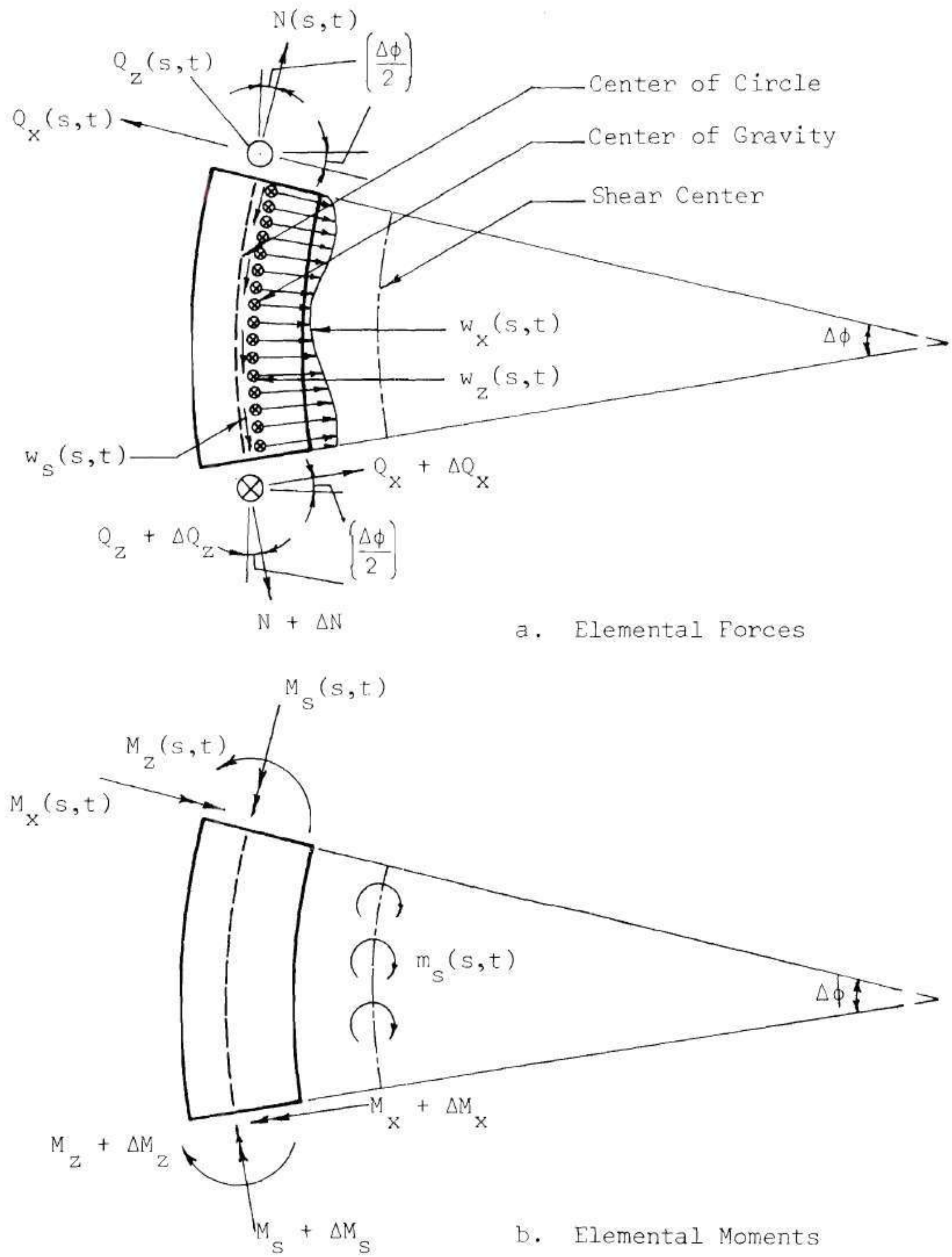

Figure 3. Elemental Forces and Moments

$$\frac{\partial}{\partial s} Q_x(s,t) + \frac{1}{R} N(s,t) + w_x(s,t) = 0$$

$$\frac{\partial}{\partial s} Q_z(s,t) + w_z(s,t) = 0 \qquad (2\text{-}1)$$

$$\frac{\partial}{\partial s} N(s,t) - \frac{1}{R} Q_x(s,t) + w_s(s,t) = 0$$

The moment equilibrium equations are

$$\frac{\partial}{\partial s} M_x(s,t) - Q_z(s,t) + \frac{1}{R} M_s(s,t) = 0$$

$$\frac{\partial}{\partial s} M_s(s,t) - \frac{1}{R} M_x(s,t) + m_s(s,t) = 0 \qquad (2\text{-}2)$$

$$\frac{\partial}{\partial s} M_z(s,t) + Q_x(s,t) = 0$$

These six equations of equilibrium can be reduced to four by elimination of Q_x and Q_y. The resulting equations are

$$-\frac{\partial^2}{\partial s^2} M_z(s,t) + \frac{1}{R} N(s,t) + w_x(s,t) = 0$$

$$\frac{\partial^2}{\partial s^2} M_x(s,t) + \frac{1}{R} \frac{\partial}{\partial s} M_s(s,t) + w_z(s,t) = 0$$

$$\frac{\partial}{\partial s} N(s,t) + \frac{1}{R} \frac{\partial}{\partial s} M_z(s,t) + w_s(s,t) = 0$$

$$\frac{\partial}{\partial s} M_s(s,t) - \frac{1}{R} M_x(s,t) + m_s(s,t) = 0 \qquad (2\text{-}3)$$

The circumferential mid-plane strain and the changes in the curvature for the extensional case are derived by Vlasov [23, pp. 448-451]. With appropriate nomenclature changes these expressions are

$$\epsilon_\phi = \left(\frac{\partial v}{\partial s} - \frac{u}{R}\right)$$

$$\chi_x = -\frac{\partial^2 w}{\partial s^2} + \frac{1}{R}\psi$$

$$\chi_z = \frac{\partial^2 u}{\partial s^2} + \frac{1}{R^2} u$$

$$\tau = \frac{\partial \psi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}$$

It is assumed that the circumferential deformation is inextensional only in the curvature expressions so that the resulting equations of motion for the in-plane vibrations will be symmetric. These relations are

$$\chi_x = -\frac{\partial^2 w}{\partial s^2} + \frac{1}{R}\psi$$

$$\chi_z = \frac{\partial^2 u}{\partial s^2} + \frac{1}{R}\frac{\partial v}{\partial s} \qquad (2\text{-}4)$$

$$\tau = \frac{\partial \psi}{\partial} + \frac{1}{R}\frac{\partial w}{\partial s}$$

Solutions based on these extensional and inextensional curvature relations have been found to agree to three decimal places for ranges of parameters under consideration in this study.

The resultant stress relations analogous to the straight beam relations are

$$N = AE\left(\frac{\partial v}{\partial s} - \frac{u}{R}\right)$$

$$M_x = -EI_x\left(\frac{\partial^2 w}{\partial s^2} - \frac{\psi}{R}\right)$$

$$M_z = EI_z\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R}\frac{\partial v}{\partial s}\right) \tag{2-5}$$

$$M_s = -EC_w\left(\frac{\partial^3 \psi}{\partial s^3} + \frac{1}{R}\frac{\partial^3 w}{\partial s^3}\right) + GI_T\left(\frac{\partial \psi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}\right)$$

where C_w is the Timoshenko warping constant [32] and is calculated for the given cross-section in Appendix A, and I_T is the St. Venant torsional constant or "effective J" expression given in Niles and Newell [33] for a thin-walled beam.

Substituting the load-displacement relations into the reduced equations of equilibrium gives the displacement-equilibrium relations

$$-EI_z\left(\frac{\partial^4 u}{\partial s^4} + \frac{1}{R}\frac{\partial^3 v}{\partial s^3}\right) + \frac{AE}{R}\left(\frac{\partial v}{\partial s} - \frac{u}{R}\right) + w_x(s,t) = 0 \tag{2-6}$$

$$- EI_x\left(\frac{\partial^4 w}{\partial s^4} - \frac{1}{R}\frac{\partial^2 \psi}{\partial s^2}\right) - \frac{EC_w}{R}\left(\frac{\partial^4 \psi}{\partial s^4} + \frac{1}{R}\frac{\partial^4 w}{\partial s^4}\right) +$$

$$\frac{GI_T}{R}\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w}{\partial s^2}\right) + w_z(s,t) = 0 \qquad (2\text{-}7)$$

$$AE\left(\frac{\partial^2 v}{\partial s^2} - \frac{1}{R}\frac{\partial u}{\partial s}\right) + \frac{EI_z}{R}\left(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R}\frac{\partial^2 v}{\partial s^2}\right) + w_s(s,t) = 0 \qquad (2\text{-}8)$$

$$- EC_w\left(\frac{\partial^4 \psi}{\partial s^4} + \frac{1}{R}\frac{\partial^4 w}{\partial s^4}\right) + GI_T\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w}{\partial s^2}\right) +$$

$$\frac{EI_x}{R}\left(\frac{\partial^2 w}{\partial s^2} - \frac{\psi}{R}\right) + m_s(s,t) = 0 \qquad (2\text{-}9)$$

When the ring vibrates freely the loads according to d'Alembert's principle [34] are of the form

$$w_x(s,t) = - \rho A \frac{\partial^2}{\partial t^2} u(s,t)$$

$$w_s(s,t) = - \rho A \frac{\partial^2}{\partial t^2} v(s,t) \qquad (2\text{-}10)$$

$$w_z(s,t) = - \rho A \frac{\partial^2}{\partial t^2} [w(s,t) - b\psi(s,t)]$$

The lateral inertia force acts through the center of gravity, but must be moved to the shear center. The final inertia torque is

$$m_s(s,t) = -\rho I_p^{c.g.} \frac{\partial^2}{\partial t^2} \psi(s,t) + \rho Ab \frac{\partial^2}{\partial t^2} [w(s,t) - b\psi(s,t)]$$

$$= -\rho I_p^{s.c.} \frac{\partial^2 \psi}{\partial t^2} + \rho Ab \frac{\partial^2 w}{\partial t^2} \tag{2-11}$$

Substituting these loads into the displacement equilibrium relations gives the eigenvalue problem

$$EI_z \left(\frac{\partial^4 u}{\partial s^4} + \frac{1}{R} \frac{\partial^3 v}{\partial s^3} \right) - \frac{AE}{R} \left(\frac{\partial v}{\partial s} - \frac{u}{R} \right) + \rho A \frac{\partial^2 u}{\partial t^2} = 0 \tag{2-12}$$

$$AE \left(\frac{\partial^2 v}{\partial s^2} - \frac{1}{R} \frac{\partial u}{\partial s} \right) + \frac{EI_z}{R} \left(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R} \frac{\partial^2 v}{\partial s^2} \right) - \rho A \frac{\partial^2 v}{\partial t^2} = 0 \tag{2-13}$$

$$EI_x \left(\frac{\partial^4 w}{\partial s^4} - \frac{1}{R} \frac{\partial^2 \psi}{\partial s^2} \right) + \frac{EC_w}{R} \left(\frac{\partial^4 \psi}{\partial s^4} + \frac{1}{R} \frac{\partial^4 w}{\partial s^4} \right) -$$

$$\frac{GI_T}{R} \left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R} \frac{\partial^2 w}{\partial s^2} \right) + \rho A \left(\frac{\partial^2 w}{\partial t^2} - b \frac{\partial^2 \psi}{\partial t^2} \right) = 0 \tag{2-14}$$

$$EC_w \left(\frac{\partial^4 \psi}{\partial s^4} + \frac{1}{R} \frac{\partial^4 w}{\partial s^4} \right) - GI_T \left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R} \frac{\partial^2 w}{\partial s^2} \right) -$$

$$\frac{EI_x}{R} \left(\frac{\partial^2 w}{\partial s^2} - \frac{\psi}{R} \right) + \rho I_p^{s.c.} \frac{\partial^2 \psi}{\partial t^2} - \rho Ab \frac{\partial^2 w}{\partial t^2} = 0 \tag{2-15}$$

These first two equations constitute the in-plane equations of motion for free vibration, while the last two are the out-of-plane or transverse equations of motion. These equations are actually two independent sets and will be treated separately in the following sections.

Out-of-Plane Vibrations

The out-of-plane equations of motion are repeated as derived above and are

$$EI_x\left(\frac{\partial^4 w}{\partial s^4} - \frac{1}{R}\frac{\partial^2 \psi}{\partial s^2}\right) + \frac{EC_w}{R}\left(\frac{\partial^4 \psi}{\partial s^4} + \frac{1}{R}\frac{\partial^4 w}{\partial s^4}\right) -$$

$$\frac{GI_T}{R}\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w}{\partial s^2}\right) + \rho A\left(\frac{\partial^2 w}{\partial t^2} - b\frac{\partial^2 \psi}{\partial t^2}\right) = 0 \tag{2-14}$$

$$EC_w\left(\frac{\partial^4 \psi}{\partial s^4} + \frac{1}{R}\frac{\partial^4 w}{\partial s^4}\right) - GI_T\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w}{\partial s^2}\right) -$$

$$\frac{EI_x}{R}\left(\frac{\partial^2 w}{\partial s^2} - \frac{\psi}{R}\right) + \rho I_p^{s.c.}\frac{\partial^2 \psi}{\partial t^2} - \rho Ab\frac{\partial^2 w}{\partial t^2} = 0 \tag{2-15}$$

These equations are developed as one matrix equation with the displacement function as the variable. An orthogonal relationship is derived for the non-degenerate modes and the response of the incomplete ring under various homogeneous boundary conditions and the complete ring subject to a general boundary is developed.

The matrix equation of motion for free vibration is

$$\rho\begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix}\begin{Bmatrix} \ddot{w}(s,t) \\ \ddot{\chi}(s,t) \end{Bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{Bmatrix} w(s,t) \\ \psi(s,t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2-16}$$

where

$$b_{11} = EI_x D_4 + \frac{EC_w}{R^2} D_4 + \frac{GI_T}{R^2} D_2$$

$$b_{12} = b_{21} = \frac{EC_w}{R} D_4 - \frac{EI_x}{R} D_2 - \frac{GI_T}{R} D_2$$

$$b_{22} = EC_w D_4 - GI_T D_2 + \frac{EI_x}{R^2}$$

and

$$D_2 = \frac{\partial^2}{\partial s^2}(\ \ ), \quad D_4 = \frac{\partial^4}{\partial s^4}(\ \ )$$

The solution for sinusoidal vibrations is

$$\begin{Bmatrix} w(s,t) \\ \psi(s,t) \end{Bmatrix} = e^{i\omega_n t}\begin{Bmatrix} W_n(s) \\ \Psi_n(s) \end{Bmatrix} \tag{2-17}$$

Substituting this solution into the equation of motion, (2-16), gives

$$-\omega_n^2\rho\begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.}\end{bmatrix}\begin{Bmatrix} W_n(s) \\ \Psi_n(s)\end{Bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{Bmatrix} W_n(s) \\ \Psi_n(s)\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2\text{-}18)$$

Premultiplying this equation by the transpose of the column displacement matrix or displacement vector, $\lfloor W_m(s) \quad \Psi_m(s) \rfloor$, and integrating over the length of the complete or incomplete ring gives

$$\int_0^{s_o} \lfloor W_m \quad \Psi_m \rfloor \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix} ds =$$

$$\rho\omega_n^2 \int_0^{s_o} \lfloor W_m \quad \Psi_m \rfloor \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.}\end{bmatrix}\begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix} ds \quad (2\text{-}19)$$

Interchanging indices and subtracting one equation from the other gives

$$\int_0^{s_o} \lfloor W_m \quad \Psi_m \rfloor \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix} ds \; -$$

$$\int_0^{s_o} \lfloor W_n \quad \Psi_n \rfloor \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{Bmatrix} W_m \\ \Psi_m \end{Bmatrix} ds =$$

$$\rho(\omega_m^2 - \omega_n^2) \int_0^{\ell} \lfloor W_m \quad \Psi_m \rfloor \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix} \begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix} ds \tag{2-20}$$

The differential operators in the [b] matrix are self-adjoint and hence the left-hand side (L.H.S.) of (2-20) can be integrated to give

$$\text{L.H.S.} = \frac{1}{2}\,[EC_w(D_1\psi_n + \frac{1}{R}\,D_1W_n)]D_2\Psi_m\Big|_0^{s_o} -$$

$$\frac{1}{2}\,[EC_w(D_1\psi_m + \frac{1}{R}\,D_1W_m)]D_2\psi_n\Big|_0^{s_o} +$$

$$[GI_T(D_1\psi_m + \frac{1}{R}\,D_1W_m) - EC_w(D_3\psi_m + \frac{1}{R}\,D_3W_m)]\psi_n\Big|_0^{s_o} -$$

$$[GI_T(D_1\psi_n + \frac{1}{R}\,D_1W_n) - EC_w(D_3\psi_n + \frac{1}{R}\,D_3W_n)]\psi_m\Big|_0^{s_o} +$$

$$[\frac{GI_T}{R}\,(D_1\psi_m + \frac{1}{R}\,D_1W_m) - \frac{EC_w}{R}\,(D_3\psi_m + \frac{1}{R}\,D_3W_m) -$$

$$EI_x(D_3W_m - \frac{1}{R}\,D_1\psi_m)]W_n\Big|_0^{s_o} -$$

$$[\frac{GI_T}{R}\,(D_1\psi_n + \frac{1}{R}\,D_1W_n) - \frac{EC_w}{R}\,(D_3\psi_n + \frac{1}{R}\,D_3W_n) -$$

$$EI_x(D_3W_n - \frac{1}{R}\,D_1\psi_n)]W_m\Big|_0^{s_o} +$$

$$\frac{EC_w}{2R}\,(D_1\psi_n + \frac{1}{R}\,D_1W_n)D_2W_m\Big|_0^{s_o} -$$

$$\frac{EC_w}{2R}\,(D_1\psi_m + \frac{1}{R}\,D_1W_m)D_2W_n\Big|_0^{s_o} +$$

$$\frac{EC_w}{2}\,(D_2\psi_m + \frac{1}{R}\,D_2W_m)D_1\psi_n\Big|_0^{s_o} -$$

$$\frac{EC_w}{2}\,(D_2\psi_n + \frac{1}{12}\,D_2W_n)D_1\psi_m\Big|_0^{s_o}$$

The L.H.S. of (2-20) is zero for incomplete rings if

either   or

$$\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R}\,\frac{\partial v}{\partial s}\right) = 0 \qquad\qquad u,_s = \bar{u},_s$$

$$\left(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R}\,\frac{\partial^2 v}{\partial s^2}\right) = 0 \qquad\qquad u = \bar{u}$$

$$AE\left(\frac{\partial v}{\partial s} - \frac{u}{R}\right) + \frac{EI_z}{R}\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R}\,\frac{\partial v}{\partial s}\right) = 0 \qquad\qquad v = \bar{v}$$

$$GI_T\left(\frac{\partial \psi}{\partial s} + \frac{1}{R}\,\frac{\partial w}{\partial s}\right) - EC_w\left(\frac{\partial^3 \psi}{\partial s^3} + \frac{1}{R}\,\frac{\partial^3 w}{\partial s^3}\right) = 0 \qquad\qquad \psi = \bar{\psi}$$

either	or
$$\left(\frac{\partial^2\psi}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w}{\partial s^2}\right) = 0$$	$$\psi,_s = \bar{\psi},_s$$
$$\left(\frac{\partial\psi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}\right) = 0$$	$$\psi,_{ss} = \bar{\psi},_{ss}$$
$$\frac{GI_T}{R}\left(\frac{\partial\psi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}\right) - \frac{EC_w}{R}\left(\frac{\partial^3\psi}{\partial s^3} + \frac{1}{R}\frac{\partial^3 w}{\partial s^3}\right) - EI_x\left(\frac{\partial^3 w}{\partial s^3} - \frac{1}{R}\frac{\partial\psi}{\partial s}\right) = 0$$	$$w = \bar{w}$$
$$EI_x\left(\frac{\partial^2 w}{\partial s^2} - \frac{\psi}{R}\right) + \frac{EC_w}{2R}\left(\frac{\partial^2\psi}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w}{\partial s^2}\right) = 0$$	$$w,_s = \bar{w},_s$$
$$\left(\frac{\partial\psi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}\right) = 0$$	$$w,_{ss} = \bar{w},_{ss}$$

The L.H.S. of (2-20) is zero for complete rings if

$$
\begin{aligned}
w(s,t) &= w(s + 2\pi R,t) \\
w'(s,t) &= w'(s + 2\pi R,t) \qquad (2\text{-}21) \\
\psi(s,t) &= \psi(s + 2\pi R,t)
\end{aligned}
$$

$$\psi'(s,t) = \psi'(s + 2\pi R,t)$$

where s is some arbitrary point or origin. With any combination of the above complete or incomplete ring conditions, equation (2-20) reduces to

$$(\omega_m^2 - \omega_n^2) \int_0^{s_o} \lfloor W_m \quad \Psi_m \rfloor \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix} \begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix} ds = 0 \qquad (2\text{-}22)$$

The polar moment of inertia with respect to an axis through the shear center can be written as

$$I_p^{s.c.} = I_G + Ab^2$$

Substituting into (2-21) gives

$$(\omega_m^2 - \omega_n^2) \int_0^{s_o} \lfloor W_m \quad \Psi_m \rfloor \begin{bmatrix} 1 & -b \\ -b & \frac{I_G}{A} + b^2 \end{bmatrix} \begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix} ds = 0$$

or

$$(\omega_m^2 - \omega_n^2) \int_0^{s_o} [(W_m - b\Psi_m)(W_n - b\Psi_n) + \frac{I_G}{A} \Psi_m \Psi_n] ds = 0 \qquad (2\text{-}23)$$

For non-degenerate modes, $\omega_m^2 \neq \omega_n^2$ for $m \neq n$ and through normalization, the orthogonal relation is

$$\frac{\int_0^{s_o} [(W_m - b\Psi_m)(W_n - b\Psi_n) + \frac{I_G}{A} \Psi_m \Psi_n] ds}{\int_0^{s_o} [(W_m - b\Psi_m)^2 + \frac{I_G}{A} \Psi_m^2] ds} = \delta_{mn} \tag{2-24}$$

where δ_{mn} is the Kronecker delta.

Note that $W_m - b\Psi_m$ is just the displacement of the center of gravity. Defining $(W_g)_m = W_m - b\Psi_m$ the orthogonality relation has the same form as the Timoshenko beam theory [35]. Namely

$$\frac{\int_0^{s_o} [(W_g)_m (W_g)_n + \frac{I_G}{A} \Psi_m \Psi_n] ds}{\int_0^{s_o} [(W_g)_m (W_g)_m + \frac{I_G}{A} \Psi_m \Psi_m] ds} = \delta_{mn} \tag{2-25}$$

The solution for general loading with homogeneous boundary conditions can be expressed as functions of the orthogonal beam vector functions. For general out-of-plane force and moment distributions the equations are

$$\rho \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix} \begin{Bmatrix} \ddot{w}(s,t) \\ \ddot{\psi}(s,t) \end{Bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} w(s,t) \\ \psi(s,t) \end{Bmatrix} = \begin{Bmatrix} F(s,t) \\ M(s,t) \end{Bmatrix} \tag{2-26}$$

Assuming a solution in the form

$$\begin{Bmatrix} W(s,t) \\ \psi(s,t) \end{Bmatrix} = \sum_{n-1}^{\infty} T_n(t) \begin{Bmatrix} W_n(s) \\ \Psi_n(s) \end{Bmatrix}$$

substituting this solution (2-27) into (2-26) and making use of (2-16) yields

$$\sum_{n=1}^{\infty}\begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix}\begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix}\ddot{T}_n(t) + \sum_{n=1}^{\infty}\begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix}\begin{Bmatrix} W_n \\ \Psi_n \end{Bmatrix}T_n(t) = \begin{Bmatrix} F(s,t) \\ M(s,t) \end{Bmatrix} \quad (2\text{-}28)$$

Premultiplying by $\lfloor W_m \quad \Psi_m \rfloor$ and using the orthogonality relation (2-25) gives

$$\ddot{T}_m(t) + \omega_m^2 T_m(t) = Q_m(t) \quad (2\text{-}29)$$

where

$$Q_m(t) = \frac{\int_0^{s_o} \lfloor W_m \quad \Psi_m \rfloor \begin{Bmatrix} F(s,t) \\ M(s,t) \end{Bmatrix} ds}{\int_0^{s_o} [(W_m - b\Psi_m)^2 + \frac{I_G}{A}\Psi_m^2] ds}$$

The solution to the above equation may be obtained by using a one-sided Green's function [36] and is

$$T_m(t) = A_m \cos\omega_m t + B_m \sin\omega_m t + \frac{1}{\omega_m}\int_0^t Q_m(\tau)\sin\omega_m(t-\tau)d\tau \quad (2\text{-}30)$$

The first portion of the solution has coefficients A_m and B_m that can be determined from initial conditions. These conditions expressed as functions of the beam vector functions are

$$\begin{Bmatrix} w(s,o) \\ \psi(s,o) \end{Bmatrix} = \begin{Bmatrix} W_o(s) \\ \psi_o(s) \end{Bmatrix} = \sum_{n=1}^{\infty} C_n \begin{Bmatrix} W_n \\ \psi_n \end{Bmatrix} \tag{2-31}$$

$$\begin{Bmatrix} \dot{w}(s,o) \\ \dot{\psi}(s,o) \end{Bmatrix} = \begin{Bmatrix} \dot{w}_o(s) \\ \dot{\psi}_o(s) \end{Bmatrix} = \sum_{n=1}^{\infty} D_n \begin{Bmatrix} w_n \\ \psi_n \end{Bmatrix}$$

From the orthogonality condition C_n and D_n are

$$C_n = \int_0^{s_o} \lfloor W_n \quad \psi_n \rfloor \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix} \begin{Bmatrix} w_o \\ \psi_o \end{Bmatrix} ds \tag{2-32}$$

$$D_n = \int_0^{s_o} \lfloor w_n \quad \psi_n \rfloor \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix} \begin{Bmatrix} w_o \\ \psi_o \end{Bmatrix} ds$$

The coefficients A_m and B_m can be solved as functions of C_m and D_m, respectively, by equating the general solution and its slope at time t=0 to the expressions (2-31) giving the solution to (2-29) in a regrouped form as

$$T_m(t) = C_m \cos\omega_m t + \frac{D_m}{\omega_m} \sin\omega_m t + \frac{1}{\omega_m} \int_0^t Q_m(\tau) \sin\omega_m (t-\tau) d\tau \tag{2-33}$$

where C_m and D_m are given in (2-23) and $Q_m(t)$ in (2-29).

Results analogous to the above have been developed by Tso [37] for the straight beam. Tso's equations can be obtained by taking the limit as $R \to \infty$ for the equations of motion (2-16), the boundary conditions (2-21), and the orthogonality conditions (2-25).

Looking at the free vibration of the complete ring, it is seen that the symmetric terms of the trigonometric Fourier series substituted into the out-of-plane ring displacement vector satisfy the appropriate boundary conditions (2-21), the orthogonality conditions (2-25), and the equations of motion (2-16). Mathematically these solutions are

$$\begin{pmatrix} W_n(s,t) \\ \psi_n(s,t) \end{pmatrix} = \begin{pmatrix} W_n \\ \Psi_n \end{pmatrix} \cos\left(\frac{ns}{R}\right) e^{i\omega_n t} \tag{2-34}$$

Substituting these solutions into (2-16) and regrouping terms results in

$$\left[\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \omega_n^2 \rho \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix} \right] \begin{pmatrix} W_n \\ \Psi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2-35}$$

where

$$K_{11} = EI_x \left(\frac{n}{R}\right)^4 + \frac{EC_w}{R^2} \left(\frac{n}{R}\right)^4 - \frac{GJ}{R^2} \left(\frac{n}{R}\right)^2$$

$$K_{12} = K_{21} = \frac{EC_w}{R} \left(\frac{n}{R}\right)^4 - \frac{EI_x}{R} \left(\frac{n}{R}\right)^2 - \frac{GJ}{R} \left(\frac{n}{R}\right)^2$$

$$K_{22} = EC_w\left(\frac{n}{R}\right)^4 - GJ\left(\frac{n}{R}\right)^2 + \frac{EI_x}{R^2}$$

The determinant of the coefficient matrix in (2-35) is the characteristic equation, that is

$$\left|\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \rho\omega_n^2 \begin{bmatrix} A & -bA \\ -bA & I_p^{s.c.} \end{bmatrix}\right| = 0 \qquad (2\text{-}36)$$

The only difference between this analysis and the results of Krahula [29], is that Krahula neglected the eccentricity of the shear center, that is, he assumed that the distance between the shear center and the center of gravity (denoted in the equations as b) was of little or no importance if b/R is a small quantity.

A third solution was obtained by Krahula by neglecting the polar moment of inertia in addition to the eccentricity b. This solution has the characteristic equation

$$|K_{11} - \omega_n^2\rho A| = 0 \qquad (2\text{-}37)$$

where K_{11} is the same as that defined in (2-35). The frequencies corresponding to this solution are given in (1-4) of Chapter I.

A fourth and final solution is the result of Michell [19]. His analysis may be deduced from the analysis of (2-36) by neglecting the warping coefficient, the polar moment of inertia, and the shear center eccentricity. This solution has the characteristic equation

$$|K'_{11} - \rho A\omega_n^2| = 0 \tag{2-37}$$

where

$$K'_{11} = EI_x\left(\frac{n}{R}\right)^4 - \frac{GJ}{R^2}\left(\frac{n}{R}\right)^2$$

The frequencies corresponding to Michell's solution are given in (1-3) of Chapter I.

These characteristic equations have been programmed on the Univac 1108 digital computer at the Rich Electronic Computer Center of the Georgia Institute of Technology. The corresponding frequencies are comparatively presented in the figures of Chapter IV along with results from the theory developed in Chapter III.

These curves show that neglecting the shear center eccentricity, b, even for b/R "small", is an invalid assumption because it results in frequencies that are prohibitively higher than the solution given in (2-36), i.e. given a system with a radii ratio of 1 to 50, the eigen-frequencies corresponding to 10 or 12 nodes will differ between the solutions with and without the shear center eccentricity by 30 to 40 per cent.

In-Plane Vibrations

The in-plane equations of motion are repeated for convenience and are

$$EI_z\left[\frac{\partial^4 u}{\partial s^4} + \frac{1}{R}\frac{\partial^3 v}{\partial s^3}\right] - \frac{AE}{R}\left[\frac{\partial v}{\partial s} - \frac{u}{R}\right] + \rho A\frac{\partial^2 u}{\partial t^2} = 0 \qquad (2\text{-}12)$$

$$AE\left[\frac{\partial^2 v}{\partial s^2} - \frac{1}{R}\frac{\partial u}{\partial s}\right] + \frac{EI_z}{R}\left[\frac{\partial^3 u}{\partial s^3} + \frac{1}{R}\frac{\partial^2 v}{\partial s^2}\right] - \rho A\frac{\partial^2 v}{\partial t^2} = 0 \qquad (2\text{-}13)$$

An orthogonality relationship and a response due to a general loading have been developed for these equations subject to a variety of boundary conditions by Lang [15]. The free vibration problem will be set up and solved in a form similar to that of the out-of-plane vibrations for completeness.

The boundary conditions require that u and v and the derivative of u be periodic. That is for the complete ring

$$u(s,t) = u(s+2\pi R,t)$$

$$u'(s,t) = u'(s+2\pi R,t) \qquad (2\text{-}38)$$

$$v(s,t) = v(s+2\pi R,t)$$

The symmetric radial terms and the antisymmetric axial terms of the trigonometric Fourier series satisfy these boundary conditions and are

$$u_n(s,t) = U_n\cos\left(\frac{ns}{R} + \theta_o\right)e^{i\omega_n t}, \; n=0,1,2,\ldots \qquad (2\text{-}39)$$

$$v_n(s,t) = V_n \sin\left(\frac{ns}{R} + \theta_o\right) e^{i\omega_n t}, \quad n=1,2,3,\ldots$$

Substituting these series into the equations of motion gives

$$\left[EI_z \left[\left(\frac{n}{R}\right)^4 U_n - \frac{1}{R}\left(\frac{n}{R}\right)^3 V_n \right] - \frac{AE}{R} \left[\left(\frac{n}{R}\right) V_n - \frac{1}{R} U_n \right] - \rho A \omega_n^2 U_n \right] \times$$

$$\cos\left(\frac{ns}{R} + \theta_o\right) e^{i\omega_n t} = 0$$

$$\left[AE \left[\left(\frac{n}{R}\right)^2 V_n - \frac{1}{R}\left(\frac{n}{R}\right) U_n \right] - \frac{EI_z}{R} \left[\left(\frac{n}{R}\right)^3 U_n - \frac{1}{R}\left(\frac{n}{R}\right)^2 V_n \right] + \rho A \omega_n^2 V_n \right] \times \tag{2-40}$$

$$\sin\left(\frac{ns}{R} + \theta_o\right) e^{i\omega_n t} = 0$$

The solutions of these equations require physical interpretation. For n=1 the solutions (2-38) are rigid body translation and the corresponding time solution is inappropriate. For n=0 the second of these equations (2-38) is identically zero. The first equation becomes

$$\left(\frac{1}{R^2} AE - \rho A \omega_o^2\right) U_o = 0$$

and its solution is

$$\omega_o^2 = \frac{E}{\rho R^2} \tag{2-41}$$

This is the natural frequency of pure radial vibration as derived by Timoshenko [38] or the so-called "breathing frequency." The final case is for $n>1$. The Equations (2-40) are satisfied for all s and t only if the coefficients of the functions of s and t are zero. The result is

$$\left[\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \omega_n^2 \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}\right] \begin{Bmatrix} U_n \\ V_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2-42}$$

$$n = 2,3,\ldots$$

where

$$K_{11} = EI_z\left(\frac{n}{R}\right)^4 + \frac{1}{R}AE$$

$$K_{12} = K_{21} = -\frac{1}{R}\left(\frac{n}{R}\right)\left\{AE + EI_z\left(\frac{n}{R}\right)^2\right\}$$

$$K_{22} = \frac{n}{R}^{2}\left[AE + \frac{1}{R^2}EI_z\right]$$

and $M_{11} = M_{22} = \rho A$.

These equations (2-41) are in turn satisfied only if the determinant of the coefficient matrix is zero. The result is the characteristic equation

$$\rho^2A^2\left(\omega_n^2\right)^2 - [EI_z\left(\frac{n}{R}\right)^2 + AE]\left[\frac{1}{R^2} + \left(\frac{n}{R}\right)^2\right]\rho A\omega_n^2 +$$

$$\left[EI_z\left(\frac{n}{R}\right)^4 + \frac{AE}{R^2}\right]\left[\frac{EI_z}{R^2} + AE\right]\left(\frac{n}{R}\right)^2 - \left[EI_z\left(\frac{n}{R}\right)^2 + AE\right]^2\left[\frac{1}{R}\left(\frac{n}{R}\right)\right]^2 = 0 \qquad (2\text{-}43)$$

This equation includes the circumferential inertia and the effect of extensionality while excluding rotary inertia and shear effects. It is henceforth regarded as the "conventional in-plane characteristic equation" and is the type equation developed by Waltking [8].

CHAPTER III

DERIVATION OF HIGHER ORDER EQUATIONS OF MOTION

Introduction

A higher order theory for the free vibrations of a thin-walled circular ring is obtained by appropriate restrictions on the deflections of the toroidal shell equations. Through the use of a variational calculus approach [39], the governing equations and natural boundary conditions are obtained. This technique was employed by Tso [37] in obtaining a higher order theory for the free vibrations of a thin-walled, open section beam, and it was used by Dzanecidze [40] in deriving Vlazov's [23] equations for the same type beam.

The higher order theory will be derived only for a slit tubular cross-section positioned so that one of the principal directions coincides with the plane of the ring. The geometry and sign convention on the deflections and independent coordinates are given in Figure 4.

The equations of motion are derived from Hamilton's principle [41] stated in mathematical form as

$$\int_{t_1}^{t_2} \int_{0}^{\theta_o} \delta(T-V)Rd\theta dt = 0 \qquad (3\text{-}1)$$

where T is the kinetic energy per circumferential length of the system and V is the strain energy per circumferential length of the system.

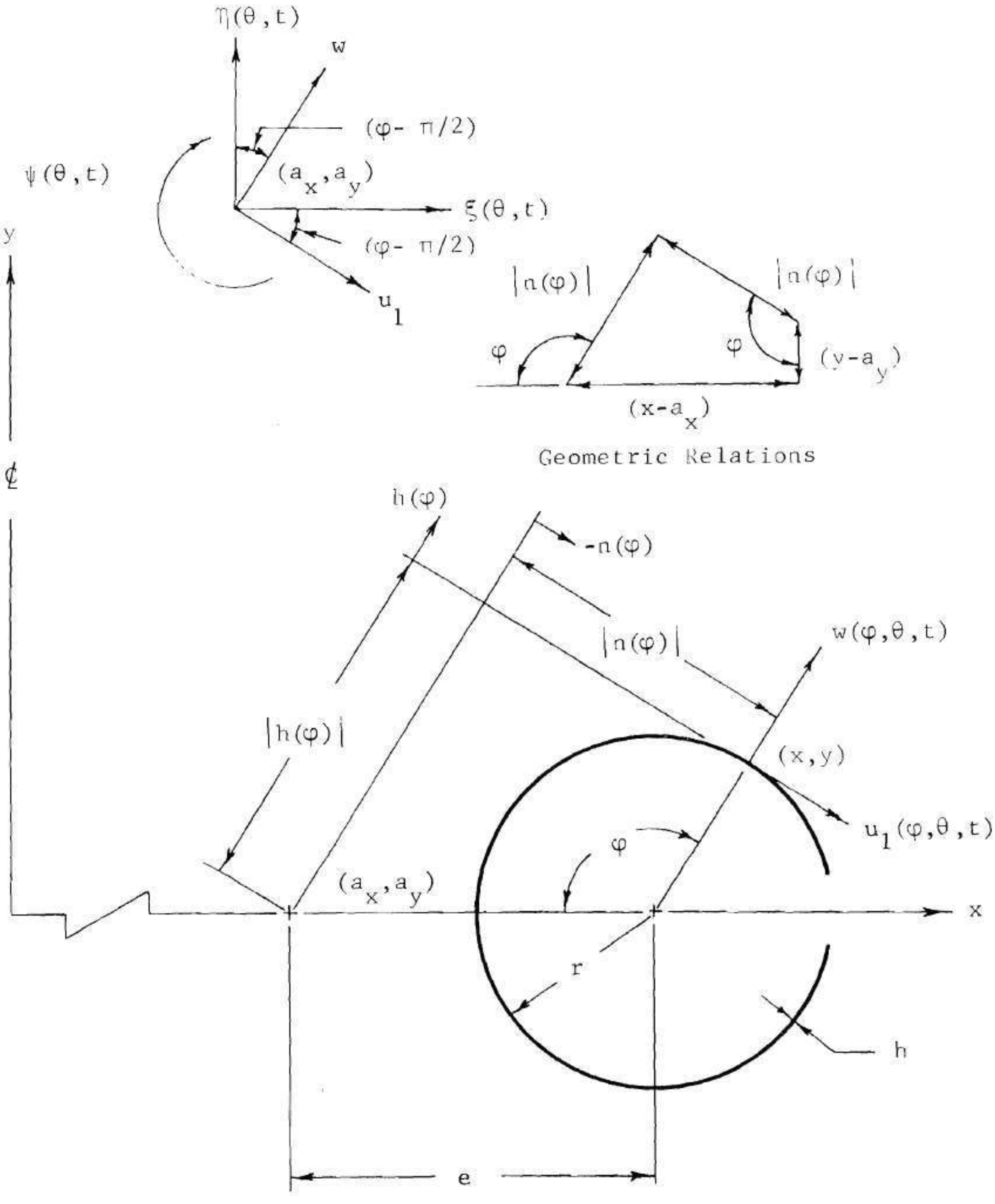

Figure 4. Geometry and Sign Convention of the Slit-Tubular Ring.

Expressions for these quantities are derived in the next section of this chapter.

The resulting equations of motion are then simplified through appropriate assumptions so as to obtain elementary theories for rings and beams.

Selection of a Suitable Shell Theory

The first problem to be confronted in this derivation is to decide which shell theory to use. Three sets of strain-displacement relations and curvature changes were considered: the theory of Novozhilov [42], Kraus' theory [43], and Sanders' linear shell theory [44]. Kraus derived his strain-displacement relations and curvature changes by placing restrictions resulting from the Kirchoff-Love assumptions on the strain displacement equations of linear elasticity as given in, say, Sokolnikoff [45]. Consequently implementation of his theory offered a derivation proceeding from linear elasticity to shell theory to ring theory. Sanders shell theory was implemented by Liepins [46] in studying the linear vibrations of a prestressed toroidal shell. Liepin's results for the eigenfrequencies and eigenmodes of axisymmetric circumferential motion of a free toroidal membrane were duplicated by McGill and Lenzen [47] through appropriate restrictions on their thick hollowed tori studies.

The most expedient (and simplest) of the coordinate systems and that employed by Novozhilov is one in which the two families of coordinate curves are simultaneously lines of principal curvature. From the first and second fundamental forms of the differential geometry these coordinates are seen to be (ϕ,θ) as given in Figure 1. Using these coordinates the Lamé parameters and the curvatures are

$$A_1 = r$$

$$A_2 = (R - r\cos\phi)$$

$$\frac{1}{R_1} = \frac{1}{r} \tag{3-2}$$

$$\frac{1}{R_2} = -\cos\phi/(R - r\cos\phi)$$

where A_1 and A_2 are the Lamé parameters and (ϕ,θ) are (α_1,α_2) of [42, p.8], respectively.

Novozhilov's resulting strain-displacement relations for the middle surface of the toroidal shell are

$$\epsilon_1 = \frac{1}{r}\left[\frac{\partial u_1}{\partial \phi} + w\right]$$

$$\epsilon_2 = \frac{1}{(R - r\cos\phi)}\left[\frac{\partial u_2}{\partial \theta} + u_1 \sin\phi - w\cos\phi\right] \tag{3-3}$$

$$\omega = \frac{(R - r\cos\phi)}{r}\frac{\partial}{\partial \phi}\left[\frac{u_2}{(R - r\cos\phi)}\right] + \frac{r}{(R - r\cos\phi)}\frac{\partial}{\partial \theta}\left[\frac{u_1}{r}\right]$$

$$= \frac{1}{r}\frac{\partial u_2}{\partial \phi} - \frac{\sin\phi}{(R - r\cos\phi)}u_2 + \frac{1}{(R - r\cos\phi)}\frac{\partial u_1}{\partial \theta}$$

where ϵ_1 is the strain in the ϕ direction

ϵ_2 is the strain in the θ or circumferential direction

ω is the in-plane shear strain

The changes in the curvature of the middle surface and the torsion of the reference surface are

$$\kappa_1 = -\frac{1}{r}\frac{\partial}{\partial\phi}\left[\frac{1}{r}\frac{\partial w}{\partial\phi} - \frac{u_1}{r}\right]$$

$$\kappa_2 = -\frac{1}{(R-r\cos\phi)^2}\left[\frac{\partial u_2}{\partial\theta}\cos\phi - \frac{\partial^2 w}{\partial\theta^2}\right] + \frac{\sin\phi}{r(R-r\cos\ )}\left[u_1 - \frac{\partial w}{\partial\phi}\right] \qquad (3\text{-}4)$$

$$\tau = \frac{1}{r(R-r\cos\phi)}\left[\frac{\partial u_1}{\partial\theta} - \frac{\partial^2 w}{\partial\phi\partial\theta} - \frac{\partial u_2}{\partial\phi}\cos\phi\right] +$$

$$\frac{1}{(R-r\cos\phi)^2}\left[\frac{\partial w}{\partial\theta}\sin\phi + u_2\sin\phi\cos\phi\right]$$

where κ_1 is the change of curvature in ϕ direction

κ_2 is the change of curvature in the θ direction

τ is the in-plane twist term

Sanders' theory used in the study of toroidal shells and Kraus' more mathematically consistent expressions differ from Novozhilov's theory only in the torsion of the reference surface term, τ. Their expressions for this term are

$$\tau_k = \frac{1}{r(R-r\cos\phi)}\left[\frac{\partial u_1}{\partial\theta} - 2\frac{\partial^2 w}{\partial\phi\partial\theta} - \frac{\partial u_2}{\partial\phi}\cos\phi + u_2\sin\phi\right] +$$

$$\frac{2}{(R-r\cos\phi)^2}\left[\frac{\partial w}{\partial \theta}\sin\phi + u_2\sin\phi\cos\phi\right]$$

$$\tau_s = \frac{1}{r(R-r\cos\phi)}\left[\frac{\partial u_1}{\partial \theta} - 2\,\frac{\partial^2 w}{\partial\phi\partial\theta} - \frac{\partial u_2}{\partial \phi}\cos\phi + u_2\sin\phi\right] +$$

$$\frac{2}{(R-r\cos\phi)^2}\left[\frac{\partial w}{\partial \theta}\sin\phi + u_2\sin\phi\cos\phi\right] +$$

$$\frac{1}{2}\left[\frac{R}{r}\,\frac{1}{(R-r\cos\phi)^2}\left[\frac{\partial u_1}{\partial \theta} - \sin\phi u_2\right] - \frac{R}{r^2(R-r\cos\phi)}\,\frac{\partial u_2}{\partial \phi}\right] \qquad (3\text{-}5)$$

where τ_k and τ_s are the torsions of the reference surface as derived by Kraus and Sanders, respectively.

It should be noted that the terms contained in the τ_k and τ_s expressions that are not contained in Novozhilov's torsion of the reference surface, τ_n, are either in-plane deflections and derivatives thereof or functions of the lateral deflection divided by the radius of the ring and will consequently contribute little to the overall strain energy. It was therefore felt that the use of Novozhilov's expressions will make the following calculations more manageable and will not differ greatly from Kraus' theory or Sanders' theory.

The strain energy density function as derived by Novozhilov is

$$U = U_\epsilon + U_\kappa$$

$$= \frac{Eh}{2(1-\nu^2)}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\nu)(\varepsilon_1\varepsilon_2 - \frac{1}{4}\,\omega^2)\right] +$$

$$\frac{Eh^3}{24(1-\nu^2)}[(\kappa_1+\kappa_2)^2 - 2(1-\nu)(\kappa_1\kappa_2-\tau^2)] \qquad (3\text{-}6)$$

The first term, U_ε, is the membrane strain energy, and the second term, U_κ, is the bending strain energy. The justifying assumption of (3-4, 5,6) are the Kirchoff-Love hypotheses and are stated here as given in Kraus [43, p.25] for completeness:

1. The shell is thin.
2. The deflections of the shell are small.
3. The transverse normal stress is negligible.
4. Normals to the reference surface of the shell remain normal to it and undergo no change in length during deformation.

Derivation of the Strain Energy Consistent with Rigid Cross-Section Displacements

The assumption that distinguishes the thin-walled curved beam from the toroidal shell is that a given cross-section of the curved beam or ring is undeformable. That is to say, a given cross-section moves as a rigid body in its own plane, but it still maintains the capability of deforming out of its plane. The rigid body motion is described by two translational coordinates and one rotational coordinate as given in Figure 4. Two additional parameters are defined in Figure 4: $n(\phi)$ and $h(\phi)$. The perpendicular distance from the origin of the beam deflection coordinates to the line of action of the lateral shell deflection is defined as $n(\phi)$. The perpendicular distance from the line of action of the in-plane shell deflection, $u_1(\phi)$, to the origin of the beam deflections is $h(\phi)$. These distances may be expressed as

$$\begin{Bmatrix} n(\phi) \\ h(\phi) \end{Bmatrix} = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{Bmatrix} x-a_x \\ y-a_y \end{Bmatrix} \tag{3-7}$$

It is further assumed that the rigid cross-section rotation of any given cross-section is small such that

$$\begin{aligned} \sin \Psi &\approx \Psi \\ \cos \Psi &\approx 1 \end{aligned} \tag{3-8}$$

Implementing the geometry of Figure 4, and the relations (3-9), the shell deflections in the plane of the cross-section as functions of the rigid cross-section or ring coordinates are

$$u_1(\phi,\theta,t) = \xi(\theta,t)\sin\phi + \eta(\theta,t)\cos\phi - \psi(\theta,t)h(\phi) \tag{3-9}$$

$$w(\phi,\theta,t) = -\xi(\theta,t)\cos\phi + \eta(\theta,t)\sin\phi + \psi(\theta,t)h(\phi) \tag{3-10}$$

Since the strain energy density function involves derivatives of the displacements u_1, u_2, and w, it will be necessary to know the derivatives of $h(\phi)$ and $n(\phi)$. Noting that

$$x(\phi) - a_x = (R - a_x) - r\cos\phi$$
$$y(\phi) - a_y = r\sin\phi - a_y \tag{3-11}$$

The first and second derivatives of $h(\phi)$ and $n(\phi)$ are

$$\begin{Bmatrix} n(\phi) \\ h(\phi) \end{Bmatrix}' = \begin{bmatrix} -\cos\phi & \sin\phi \\ -\sin\phi & -\cos\phi \end{bmatrix} \begin{Bmatrix} x-a_x \\ y-a_y \end{Bmatrix} - \begin{Bmatrix} r \\ 0 \end{Bmatrix} \tag{3-12}$$

$$\begin{Bmatrix} n(\phi) \\ h(\phi) \end{Bmatrix}'' = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix} \begin{Bmatrix} x-a_x \\ y-a_y \end{Bmatrix} - \begin{Bmatrix} 0 \\ r \end{Bmatrix}$$

A convenient representation of the rigid cross-section or ring deflections is obtained by separating these deflections into two parts: the deformation caused by neglecting the shear strain in the approximate strain energy density expression, subscripted "b"; and the deformation due to that shear strain in the strain energy density expression, subscripted "s". That is

$$\xi(\theta,t) = \xi_b(\theta,t) + \xi_s(\theta,t)$$

$$\eta(\theta,t) = \eta_b(\theta,t) + \eta_s(\theta,t) \tag{3-13}$$

$$\psi(\theta,t) = \psi_b(\theta,t) + \psi_s(\theta,t)$$

Carrying this convention through to the original deflections gives

$$u_2 = u_{2_b} + u_{2_s}$$

$$u_1 = u_{1_b} + u_{1_s}$$

$$= (\xi_b+\xi_s)\sin\phi + (\eta_b+\eta_s)\cos\phi - (\psi_b+\psi_s)h(\phi) \qquad (3\text{-}14)$$

$$w = w_b + w_s$$

$$= -(\xi_b+\xi_s)\cos\phi + (\eta_b+\eta_s)\sin\phi + (\psi_b+\psi_s)n(\phi)$$

The above convention is such that u_{2_b} can be determined as a function of the rigid cross-section coordinates ξ_b, u_b, and Ψ_b by noting that the b-subscripted terms are for zero shear strain. Splitting the shear strain into two parts such that

$$\omega = \omega_b + \omega_s \qquad (3\text{-}15)$$

and defining ω_b such that

$$\omega_b = 0 \qquad (3\text{-}16)$$

leads to the equation

$$\omega_b = \left(\frac{R-r\cos\phi}{r}\right) \frac{\partial}{\partial\phi} \left(\frac{u_{2_b}}{R-r\cos\phi}\right) +$$

$$\left(\frac{r}{R-r\cos\phi}\right) \frac{\partial}{\partial\theta} \left(\frac{u_{1_b}}{r}\right) = 0 \qquad (3\text{-}17)$$

From this relation (3-17), $u_{2_b}(\theta,\phi,t)$ can be determined as a function of the rigid cross-section coordinates. Substituting the expression for $u_{1_b}(\theta,\phi,t)$ from (3-14) into (3-17) and solving for $u_{2_b}(\theta,\phi,t)$ gives

$$u_{2_b}(\theta,\phi,t) = \left(\frac{R-r\cos\phi}{R-r\cos\phi_a}\right) u_{2_b}(\theta,\phi_a,t) -$$

$$(R-r\cos\phi) \int_{\phi_a}^{\phi} \frac{r[\xi_b'\sin\bar{\phi} + \eta_b'\cos\bar{\phi} - \psi_b' h(\bar{\phi})]}{[R-r\cos\bar{\phi}]^2} d\bar{\phi} \qquad (3\text{-}18)$$

where $u_{2_b}(\theta,\phi_a,t)$ is the circumferential deflection at the point where $\phi=\phi_a$ and other parameters are as defined.

Before evaluating the integral in (3-18) it is advantageous from an algebraic point of view to restrict the origin of the rigid cross-section coordinates to coincide with the shear center of a straight beam with the same cross-section as the ring. Doing so makes many subsequent integrals symmetric or antisymmetric over even intervals, that is with limits of $\pm\alpha$.

Under this restriction (3-11) becomes

$$x(\phi) - a_x = e - r\cos\phi \qquad (3\text{-}19)$$

$$y(\phi) - a_y = r\sin\phi$$

where

$$e = 2r\left[\frac{\sin\phi_o + (\pi-\phi_o)\cos\phi_o}{(\pi-\phi_o) + \sin\phi_o\cos\phi_o}\right]$$

and e is the distance from the shear center to the center of the circular cross-section.

Consequently $h(\phi)$ can be expressed as

$$h(\phi) = e\cos\phi - r \tag{3-20}$$

Substituting (3-20) into the integral of (3-18) and rearranging gives

$$\int_{\phi_a}^{\phi} \frac{r[\xi_b'\sin\bar{\phi} + \eta_b'\cos\bar{\phi} - \psi_b'h(\bar{\phi})]}{[R-r\cos\bar{\phi}]^2}\, d\bar{\phi} =$$

$$r\xi_b'(\theta,t)\int_{\phi_a}^{\phi} \frac{\sin\bar{\phi}}{[R-r\cos\bar{\phi}]^2}\, d\bar{\phi} + r^2\psi_b'(\theta,t)\int_{\phi_a}^{\phi} \frac{d\bar{\phi}}{[R-r\cos\bar{\phi}]^2} +$$

$$r[\eta_b'(\theta,t) - e\psi_b'(\theta,t)]\int_{\phi_a}^{\phi} \frac{\cos\bar{\phi}}{[R-r\cos\bar{\phi}]^2}\, d\bar{\phi} \tag{3-21}$$

The first integral in the R.H.S. of (3-21) may be easily evaluated and is

$$\int_{\phi_a}^{\phi} \frac{\sin\bar{\phi}\,d\bar{\phi}}{(1-a\cos\bar{\phi})^2} = \left.\frac{1}{Rr(1-a\cos\bar{\phi})}\right|_{\phi_a}^{\phi} \tag{3-22}$$

where a is the ratio of radii, r/R.

The second and third integrals on the R.H.S. of (3-21) require the substitution

$$z = \tan(\phi/2)$$

for evaluation since the integral is a rational function of sines and cosines. The results are

$$\frac{1}{R^2}\int_{\phi_a}^{\phi} \frac{\cos\bar{\phi}\,d\bar{\phi}}{(1-a\cos\bar{\phi})^2} = \frac{2}{R^2}\,\frac{1}{(1+a)^2}\left[\frac{(1+a)\tan(\bar{\phi}/2)}{(1-a)[(1-a)+(1+a)\tan^2(\bar{\phi}/2)]} + \right.$$

$$\left.\left.\frac{a}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left\{\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\bar{\phi}/2)\right]\right\}\right]\right|_{\phi_a}^{\phi}$$

$$\frac{1}{R^2}\int_{\phi_a}^{\phi} \frac{d\bar{\phi}}{(1-a\cos\bar{\phi})^2} = \frac{2}{R^2}\,\frac{1}{(1+a)^2}\left[\frac{(1+a)a\tan(\bar{\phi}/2)}{(1-a)[(1-a)+(1+a)\tan^2(\bar{\phi}/2)]} + \right.$$

$$\left.\left.\frac{1}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left\{\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\bar{\phi}/2)\right]\right\}\right]\right|_{\phi_a}^{\phi} \tag{3-24}$$

Substituting (3-22), (3-23) and (3-24) into (3-21), choosing ϕ_a to be zero and substituting this result into (3-18) gives the expression for

u_{2_b} an an explicit function of θ, ϕ, and t. It is

$$u_{2_b}(\theta,\phi,t) = \frac{(1-a\cos\phi)}{(1-a)}\zeta(\theta,t) - \left[1 - \frac{(1-a\cos\phi)}{(1-a)}\right]\xi_b'(\theta,t) -$$

$$\frac{2a}{(1+a)^2}(1-a\cos\phi)\left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a)+(1+a)\tan^2(\phi/2)]} + \right.$$

$$\left.\left(\frac{a}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\right][\eta_b'(\theta,t) - e\psi_b'(\theta,t)] -$$

$$\frac{2a}{(1+a)^2}(1-a\cos\phi)\left[\frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a)+(1+a)\tan^2(\phi/2)]} + \right.$$

$$\left.\left(\frac{1}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\right]r\psi_b'(\theta,t)$$

where $\zeta(\theta,t) = u_{2_b}(\theta,\phi_a,t)\Big|_{\phi_a=0}$.

The circumferential deflection due to the shear, $u_{2_s}(\theta,\phi,t)$, cannot be determined as a function of the ring deflections. This deflection is small in comparison to $u_{2_b}(\theta,\phi,t)$, however, and is neglected in the following analysis. That is,

$$u_2(\theta,\phi,t) = u_{2_b}(\theta,\phi,t) + u_{2_s}(\theta,\phi,t)$$

$$\approx u_{2_b}(\theta,\phi,t)$$

where $u_{2_b}(\theta,\phi,t)$ is the expression given in (3-25).

Substituting the expressions for the shell deflections u_1, u_2, and w as functions of the rigid cross-section deflections ξ_b, ξ_s, η_b, η_s, Ψ_b, Ψ_s, and ζ into the strain displacement relations reveals that

$$\begin{aligned} \varepsilon_1 &= 0 \\ \kappa_1 &= 0 \end{aligned} \qquad (3\text{-}27)$$

This is expected for the maintenance of a rigid cross-section. Given a thin-walled ring with a relatively small thickness, the circumferential membrane effect will dominate the circumferential bending effect. That is

$$\kappa_2 h << \varepsilon_2$$

and this bending effect will be dropped from further consideration. It is therefore possible to write the strain energy density expression as

$$U = \frac{Eh}{2(1-\nu^2)}\left[\varepsilon_2^2 + \left(\frac{1-\nu}{2}\right)\omega^2\right] + \frac{Eh^3}{12(1+\nu)}\tau^2 \qquad (3\text{-}28)$$

Substituting the strain displacement relations (3-3) into (3-28) gives the strain energy density as a function of the shell displacements

$$U = \frac{Eh}{2(1-\nu^2)}\left[\frac{1}{(R-r\cos\phi)}\left(\frac{\partial u_2}{\partial\phi} + u_1\sin\phi - w\cos\phi\right)^2 + \right.$$

$$\left(\frac{1-\nu}{2}\right)\left(\frac{1}{r}\frac{\partial u_2}{\partial\phi} - \frac{\sin\phi}{(R-r\cos\phi)}u_2 + \frac{1}{(R-r\cos\phi)}\frac{\partial u_1}{\partial\theta}\right)^2\Bigg] +$$

$$\frac{Eh^3}{12(1+\nu)}\Bigg[\frac{1}{r(R-r\cos\phi)}\left(\frac{\partial u_1}{\partial\theta} - \frac{\partial u_2}{\partial\phi}\cos\phi - \frac{\partial^2 w}{\partial\theta\partial\phi}\right) +$$

$$\frac{1}{(R-r\cos\phi)^2}\left(\frac{\partial w}{\partial\theta}\sin\phi + u_2\sin\phi\cos\phi\right)\Bigg]^2 \tag{3-29}$$

The expressions for u_2, u_1, and w, **equations** (3-26) and (3-10), respectively, can be substituted into (3-29) to give the strain energy density as a function of the rigid cross-sectional deflections. This expression must be integrated in turn over the surface of the ring to give the strain energy per circumferential length. That is

$$V(\theta,t) = \int_{-\alpha}^{\alpha} U(\theta,\bar{\phi},t)(1-a\cos\bar{\phi})Rd\bar{\phi} \tag{3-30}$$

where $\alpha = \pi - \phi_o$. The integration in (3-30) is extremely intricate and requires some form of approximate integration.

Strain Energy Expression Using Simpson's Rule

The first numerical integration technique is a solution using Simpson's one-third rule of the Newton-Cotes Formulae [48]. It is

$$\int_a^b f(x)dx = \frac{\Delta}{3}(f_o + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots$$

$$\ldots + 2f_{N-2} + 4f_{N-1} + f_N) - \frac{1}{180}(b-a)\Delta^4 f^{(4)}(\xi) \tag{3-31}$$

where $a<\xi<b$ and Δ is the increment between evaluated points. In this solution that increment Δ was chosen as one degree around the arc; the error is consequently insignificant. The resulting strain energy expression is

$$V(\theta,t) = \frac{E}{(1-\nu^2)}\left[I_1 \frac{1}{R^2}\zeta'^2(\theta,t) + I_2 \frac{1}{R^4}\xi_b''^2(\theta,t) + I_3 \frac{1}{R^4}(\eta_b''-e\psi_b'')^2 + \right.$$

$$I_4\left(\frac{r}{R^2}\psi_b''\right)^2 + I_5 \frac{2}{R}\zeta' \frac{1}{R^2}\xi_b'' + I_6 \frac{2r}{R^4}\psi_b''(\eta_b''-e\psi_b'') +$$

$$I_7(\xi_b+\xi_s)^2 + I_8 r^2(\psi_b+\psi_s)^2 + I_9 \frac{2}{R}\zeta'(\xi_b+\xi_s) +$$

$$I_{10}\frac{2}{R^2}\xi_b''(\xi_b+\xi_s) - I_{11}\frac{2r}{R^2}(\psi_b+\psi_s) \times \left(\eta_b''-e\psi_b''\right) -$$

$$I_{12}2\left(\frac{r}{R}\right)^2\psi_b''(\psi_b+\psi_s)] + G\left[I_{13}\left(\frac{1}{R}\xi_s'\right)^2 + I_{14}\frac{1}{R}(\eta_s'-e\psi_s')^2 + \right.$$

$$I_{15}\left(\frac{r}{R}\psi_s'\right)^2 + I_{16}2\left(\frac{r}{R}\right)\psi_s' \frac{1}{R}(\eta_s'-e\psi_s')\right] + \frac{G}{3}\left[I_{17}\left(\frac{r}{R}\right)^2(\psi_b'+\psi_s')^2 + \right.$$

$$I_{18}\zeta^2(\theta,t) + I_{19}\frac{1}{R^2}\xi_b'^2(\theta,t) + I_{20}\frac{1}{R^2}(\eta_b'-e\psi_b')^2 -$$

$$I_{21}\frac{2}{R}\zeta(\theta,t)\xi_b'(\theta,t) + I_{22}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) + I_{23}\frac{2r}{R^2}\psi_b'(\eta_b'-e\psi_b') +$$

$$I_{24}\frac{1}{R^2}\xi_b'^2(\theta,t) + I_{25}\frac{1}{R^2}(\eta_b'-e\psi_b')^2 + I_{26}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) +$$

$$I_{27}\left(\frac{2r}{R^2}\right)\psi_6'(\eta_b'-e\psi_b') + I_{28}\left(\frac{2r}{R^2}\right)(\psi_b'+\psi_s')(\eta_b'-e\psi_b') +$$

$$I_{29}\left(2\,\frac{r^2}{R^2}\right)\psi_b'(\psi_b'+\psi_s') + I_{30}\,\frac{2r}{R^2}(\psi_b'+\psi_s')(\eta_b'-e\psi_b') + I_{31}2\left(\frac{r}{R}\right)^2\psi_b'(\psi_b'+\psi_s') -$$

$$I_{32}\,\frac{2}{R}\,\zeta(\theta,t)\xi_b'(\theta,t) + I_{33}\,\frac{2}{R^2}\,\xi_b'^2(\theta,t) + I_{34}\,\frac{2}{R^2}\,(\eta_b'-e\psi_b')^2 +$$

$$I_{35}\left(\frac{2r}{R^2}\right)\psi_b'(\eta_b'-e\psi_b') + I_{36}\left(\frac{2r}{R^2}\right)\psi_b'(\eta_b'-e\psi_b') + I_{37}2\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) +$$

$$I_{18}\zeta^2(\theta,t) + I_{19}\,\frac{1}{R^2}\,\xi_b'^2(\theta,t) + I_{20}\,\frac{1}{R^2}(\eta_b'-e\psi_b')^2 +$$

$$I_{22}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) - I_{21}\,\frac{2}{R}\,\zeta(\theta,t)\xi_b'(\theta,t) + I_{23}\left(\frac{2r}{R^2}\right)\psi_b'(\eta_b'-e\psi_b') +$$

$$I_{24}\,\frac{1}{R^2}\,(\xi_b'+\xi_s')^2 + I_{38}\,\frac{1}{R^2}\,[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')]^2 -$$

$$I_{32}\,\frac{2}{R}\,\zeta(\theta,t)(\xi_b'+\xi_s') + I_{33}\,\frac{2}{R^2}\,\xi_b'(\xi_b'+\xi_s') - I_{39}2(\eta_b'-e\psi_b')[(\eta_b'+\eta_s') -$$

$$e(\psi_b'+\psi_s')] - I_{40}\left(\frac{2r}{R^2}\right)\psi_b'[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] +$$

$$2\left[-I_{41}\left(\frac{r}{R}\right)^2(\psi_b'+\psi_s')(\eta_b'-e\psi_b') - I_{42}\left(\frac{r}{R}\right)^2\psi_b'(\theta,t)(\psi_b'+\psi_s') +\right.$$

$$I_{43}\left(\frac{r}{R^2}\right)(\psi_b'+\psi_s')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] - I_{18}\zeta^2(\theta,t) +$$

$$I_{21}\,\frac{1}{R}\,\xi_b'(\theta,t)\zeta(\theta,t) + I_{32}\,\frac{1}{R}\,(\xi_b'+\xi_s')\zeta(\theta,t) + I_{21}\,\frac{1}{R}\,\zeta(\theta,t)\xi_b'(\theta,t) -$$

$$I_{19}\,\frac{1}{R^2}\,\xi_b'^{\,2}(\theta,t) - I_{33}\,\frac{1}{R^2}\,(\xi_b'+\xi_s')\xi_b'(\theta,t) - I_{20}\,\frac{1}{R^2}\,(\eta_b'-e\psi_b')^2 -$$

$$I_{13}\,\frac{r}{R^2}\,\psi_b'(\theta,t)(\eta_b'-e\psi_b') + I_{39}\,\frac{1}{R^2}\,(\eta_b'-e\psi_b')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] -$$

$$I_{23}\,\frac{r}{R^2}\,\psi_b'(\theta,t)(\eta_b'-e\psi_b') - I_{22}\left(\frac{r}{R}\right)^2\psi_b'^{\,2}(\theta,t) + I_{40}\left(\frac{r}{R^2}\right)\psi_b'[(\eta_b'+\eta_s') -$$

$$e(\psi_b'+\psi_s')] + I_{32}\,\frac{1}{R}\,\zeta(\theta,t)\xi_b'(\theta,t) - I_{33}\,\frac{1}{R^2}\,\xi_b'^{\,2}(\theta,t) - I_{34}\,\frac{1}{R^2}\,(\eta_b'-e\psi_b')^2 -$$

$$I_{36}\,\frac{r}{R^2}\,\psi_b'(\theta,t)(\eta_b'-e\psi_b') - I_{35}\,\frac{r}{R^2}\,\psi_b'(\theta,t)(\eta_b'-e\psi_b') - I_{37}\left(\frac{r}{R}\right)^2\psi_b'^{\,2}(\theta,t) -$$

$$I_{24}\,\frac{1}{R^2}\,\xi_b'(\theta,t)(\xi_b'+\xi_s') + I_{24}\left(\frac{1}{R^2}\right)(\eta_b'-\psi_b')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] +$$

$$I_{44}\left(\frac{r}{R^2}\right)\psi_b'(\theta,t)[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')]\Bigg]\Bigg] \qquad (3\text{-}32)$$

where the coefficients are defined in Appendix B.

A second approximate integration of (3-30) is given in the following section.

Strain Energy Expression Using Binomial Series Expansion of the Integrand

The integral in (3-30) cannot be evaluated explicitly if the (3-25) version of $u_2(\theta,\phi,t)$ is implemented, but an alternate approximate derivation of this expression can be used in (3-30) to obtain the strain energy.

The ratio of the radius of the ring cross-section to the radius of the ring, a, must be a reasonably small number for the assumption that the ring cross-section does not deform in its own plane to hold. This ratio is assumed to be less than or equal to 0.20. That is

$$a = \frac{r}{R} \leq \frac{1}{5}$$

With this in mind, the integrals of (3-23) and (3-24) can be approximated by expanding the integrands in a binomial series expansion and integrating the results. The expansion of the denominator of (3-23) and (3-24) is

$$\frac{1}{(1-a\cos\phi)^2} = \sum_{n=0}^{\infty} (n+1)a^n\cos^n\phi = f(\phi) \tag{3-33}$$

for $-\pi \leq \phi \leq \pi$

But $|(n+1)a^n\cos^n\phi| \leq (n+1)a^n$, and by d'Alembert's ratio test the series

$$\sum_{n=0}^{\infty} (n+1)a^n < \infty$$

Therefore by the Weierstrass M-test, the series

$$\sum_{n=0}^{\infty} (n+1)a^n\cos^n\phi$$

converges uniformly in $-\pi \leq \phi \leq \pi$. Multiplying this series termwise by the continuous function $\cos\phi$ gives the uniformly convergent series

$$\cos\phi f(\phi) = \sum_{n=0}^{\infty} (n+1)a^n \cos^{n+1}\phi \tag{3-34}$$

Hence termwise integration is possible and it is permissible to write (3-23) and (3-24) in the forms

$$\frac{1}{R^2}\int_{\phi_a}^{\phi} \frac{\cos\bar{\phi}d\bar{\phi}}{(1-a\cos\bar{\phi})^2} = \frac{1}{R^2}\sum_{n=0}^{\infty}\int_{\phi_a}^{\phi} (n+1)a^n\cos^{(n+1)}\bar{\phi}d\bar{\phi} \tag{3-35}$$

$$\frac{1}{R^2}\int_{\phi_a}^{\phi} \frac{d\bar{\phi}}{(1-a\cos\bar{\phi})^2} = \frac{1}{R^2}\sum_{n=0}^{\infty}\int_{\phi_a}^{\phi} (n+1)a^n\cos^{n}\bar{\phi}d\bar{\phi} \tag{3-36}$$

These integrals may be evaluated termwise [49] and a term integration is

$$\int_{\phi_a}^{\phi} (n+1)a^n\cos^{(n+1)}\bar{\phi}d\bar{\phi} = \left[\sin\phi \sum_{\nu=1}^{\mu} \frac{(n;-2;\nu-1)}{(n+1;-2;\nu)}\cos^{n+2(1-\nu)}\bar{\phi} + \frac{(1-s;2;\mu)}{(2-s;2;\mu)}\right]\Bigg|_{\phi_a}^{\phi} \tag{3-37}$$

with $n+1 = 2\mu-s$, $s=0$ or 1.

A similar expression is obtained for a single term integration of (3-36). Hence the integrals (3-23) and (3-24) may be expressed as

$$\frac{1}{R^2}\int_{\phi_a}^{\phi} \frac{\cos\bar{\phi}d\bar{\phi}}{(1-a\cos\bar{\phi})^2} = \frac{1}{R^2}\sum_{n=0}^{\infty} (n+1)a^n \times$$

$$\left[\sin\bar{\phi}\sum_{\nu=1}^{\mu}\frac{(n;-2;\nu-1)}{(n+1;-2;\nu)}\cos^{[n+2(1-\nu)]}\bar{\phi}+\frac{(1-s;2;\mu)}{(2-s;2;\mu)}\right]\Bigg|_{\phi_a}^{\phi} \tag{3-38}$$

and

$$\frac{1}{R^2}\int_{\phi_2}^{\phi}\frac{d\bar{\phi}}{(1-a\cos\bar{\phi})^2}=\frac{1}{R^2}\sum_{n=0}^{\infty}(n+1)a^n\times$$

$$\left[\sin\bar{\phi}\sum_{\nu=1}^{\mu}\frac{(n-1;-2;\nu-1)}{(n;-2;\nu)}\cos^{(n-2\nu+1)}\bar{\phi}+\frac{(1-s;2;\mu)}{(2-s;2;\mu)}\right]\Bigg|_{\phi_a}^{\phi} \tag{3-39}$$

where $(m;-d;\nu) = d^2\Gamma\left(\frac{m}{d}+1\right)/\Gamma\left(\frac{m}{d}-\nu+1\right)$ with $(n+1) = 2\mu - s$, $s = 0$ or 1.

Practical ratios of the radii in transport type aircraft are usually less than or equal to 1/25. In this paper, as stated earlier, the ratio is assumed to be less than or equal to 1/5. Hence a reasonable approximation of (3-38) and (3-39) may be obtained by truncating all terms in the series containing terms a^n where $n>2$. These approximations with $\phi_a = 0$ are

$$\frac{1}{R^2}\int_0^{\phi}\frac{\cos\bar{\phi}d\bar{\phi}}{(1-a\cos\bar{\phi})^2}\simeq\frac{1}{R^2}\left[\sin\phi+a(\phi+\frac{1}{2}\sin2\phi)+a^2\sin\phi(2+\cos^2\phi)\right] \tag{3-40}$$

$$\frac{1}{R^2}\int_0^{\phi}\frac{d\bar{\phi}}{(1-a\cos\bar{\phi})^2}\simeq\frac{1}{R^2}[\phi+2a\sin\phi+\frac{3}{2}a^2(\phi+\frac{1}{2}\sin2\phi)] \tag{3-41}$$

Equations (3-40) and (3-41) have been compared with the "exact results" in (3-23) and (3-24). Table 1 shows comparisons between these expressions for $\phi_o = 45°$ and for values of r/R from 0 to 0.4. Column A is based on the "exact" expression (3-23); column B represents the approximate counterpart of (3-23), equation (3-40); column C is the "exact" equation (3-24), and column D is the approximate equation (3-41). The results are seen to be quite good for radii ratios of 0.2 or below. In fact comparisons for ϕ_o ranging from 1° to 135° have been made, and an upper bound on the error for a radii ratio of 0.20 has been determined as 4 per cent.

Substituting (3-22) for $\phi_a = 0$ and (3-40) and (3-41) into (3-21) and in turn into (3-18) gives an approximate expression for u_{2_b} as

$$u_{2_b}(\theta,\phi,t) = \left(\frac{1 - a\cos\phi}{1 - a}\right)\zeta(\theta,t) - (R - r\cos\phi)[\xi_b'(\theta,t) \times$$

$$\left[\frac{1}{(R - r)} - \frac{1}{(R - r\cos\phi)}\right] + \frac{r}{R^2}[\eta_b'(\theta,t) - e\psi_b'(\theta,t)] \times$$

$$[\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(2 + \cos^2\phi)] +$$

Table 1. Comparisons of Exact versus Approximate Pertinent Integrals for $\phi_o = 45°$

a	A	B	C	D
0.40	2.013	1.732	3.655	3.367
0.39	1.955	1.700	3.593	3.331
0.38	1.900	1.668	3.533	3.296
0.37	1.846	1.636	3.475	3.261
0.36	1.795	1.604	3.420	3.226
0.35	1.745	1.573	3.367	3.192
0.34	1.698	1.543	3.316	3.159
0.33	1.652	1.512	3.267	3.126
0.32	1.607	1.482	3.220	3.094
0.31	1.564	1.452	3.175	3.062
0.30	1.523	1.423	3.131	3.031
0.29	1.483	1.394	3.090	3.000
0.28	1.444	1.365	3.049	2.970
0.27	1.407	1.337	3.011	2.941
0.26	1.370	1.309	2.973	2.912
0.25	1.335	1.282	2.937	2.884
0.24	1.301	1.254	2.902	2.856
0.23	1.268	1.228	2.869	2.829
0.22	1.236	1.201	2.837	2.802
0.21	1.205	1.175	2.806	2.776
0.20	1.175	1.149	2.776	2.750
0.19	1.145	1.124	2.747	2.725
0.18	1.117	1.098	2.719	2.701
0.17	1.089	1.074	2.692	2.677
0.16	1.062	1.049	2.666	2.654
0.15	1.035	1.025	2.641	2.631
0.14	1.010	1.002	2.617	2.609
0.13	0.965	0.978	2.593	2.587
0.12	0.960	0.955	2.571	2.566
0.11	0.937	0.933	2.549	2.545
0.10	0.913	0.910	2.528	2.525
0.09	0.891	0.889	2.508	2.506
0.08	0.868	0.867	2.488	2.487
0.07	0.847	0.846	2.470	2.469
0.06	0.825	0.825	2.452	2.452
0.05	0.805	0.804	2.434	2.434
0.04	0.784	0.784	2.417	2.417
0.03	0.765	0.764	2.401	2.401
0.02	0.745	0.745	2.386	2.386
0.01	0.726	0.726	2.371	2.371
0.00	0.707	0.707	2.356	2.356

$$\left(\frac{r}{R}\right)^2 \psi_b'(\theta,t)[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)] \tag{3-42}$$

Expressing the strain energy density as a function of the rigid cross-section deformations requires knowing the derivative of the shell deformation variable $u_{2_b}(\theta,\phi,t)$ with respect to ϕ. This derivative is

$$\frac{r}{R^2}\frac{\partial}{\partial\phi}\left(\int_{\phi_2}^{\phi}\frac{[\xi_b'(\theta,t)\sin\bar{\phi} + \eta_b'(\theta,t)\cos\bar{\phi} - \psi_b'(\theta,t)h(\bar{\phi})]}{(1 - a\cos\bar{\phi})^2}\,d\bar{\phi}\right) =$$

$$\frac{r}{R^2}\frac{1}{(1 - a\cos\phi)^2}[\xi_b'\sin\phi + \eta_b'\cos\phi - \psi_b'h(\phi)] \tag{3-43}$$

Hence the derivative of $u_{2_b}(\theta,\phi,t)$ with respect to ϕ may be written as

$$\frac{\partial}{\partial\phi}\left(u_{2_b}(\theta,\phi,t)\right) = r\sin\phi\left[\frac{1}{(R-r)}\ \zeta(\theta,t) -\right.$$

$$\left[\xi_b'(\theta,t)\left[\frac{1}{(R-r)} - \frac{1}{(R-r\cos\phi)}\right] + \frac{r}{R^2}[\eta_b'(\theta,t) - e\psi_b'(\theta,t)] \times\right.$$

$$[\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(2 + \cos^2\phi)] +$$

$$\left(\frac{r}{R}\right)^2 \psi_b'(\theta,t)[\phi + 2a\sin\phi + \frac{3}{2} a^2(\phi + \frac{1}{2}\sin 2\phi)]\Bigg]\Bigg] -$$

$$\frac{1}{(R-r\cos\phi)} [r\xi_b'(\theta,t)\sin\phi + r(\eta_b'(\theta,t) - e\psi_b'(\theta,t))\cos\phi +$$

$$r^2\psi_b'(\theta,t)] \qquad (3\text{-}44)$$

Substituting the shell displacements u_1, u_2, and w and their derivatives, as functions of the rigid cross-section coordinates, into the strain energy density function (3-29) gives the relation

$$U(\theta,\phi,t) = \frac{Eh}{2(1-\nu^2)} \left[\left[\left[\frac{1}{(R-r)} \zeta'(\theta,t) - \left[\frac{1}{(R-r)} - \frac{1}{(R-r\cos\phi)}\right]\xi_b''(\theta,t) -\right.\right.\right.$$

$$[\sin\phi + a\phi + \frac{1}{2} a\sin 2\phi + a^2\sin\phi(\cos^2\phi+2)] \frac{r}{R^2}[\eta_b''(\theta,t) - e\psi_b''(\theta,t)] -$$

$$[\phi + 2a\sin\phi + \frac{3}{2} a^2(\phi + \frac{1}{2}\sin 2\phi)]\left(\frac{r}{R}\right)^2 \psi_b''(\theta,t) +$$

$$\left.\left.\frac{1}{(R-r\cos\phi)} [(\xi_b+\xi_s) + r(\psi_b+\psi_s)\sin\phi]\right]\right]^2 + \frac{1-\nu}{2}\left[\left[\frac{1}{(R-r\cos\phi)} [\xi_s'\sin\phi +\right.\right.$$

$$\left.\left.\eta_s'\cos\phi - \psi_s'h(\phi)]\right]^2\right] + \frac{1}{2} G \frac{h^3}{3}\left[\frac{1}{r(R-r\cos\phi)}\left[r(\psi_b'+\psi_s') -\right.\right.$$

$$r\sin\phi\cos\phi\left[\frac{1}{(R-r)}\zeta(\theta,t) - \xi_b'(\theta,t)\left[\frac{1}{(R-r)} - \frac{1}{(R-r\cos\phi)}\right] -\right.$$

$$\frac{r}{R^2}\,[\eta_b'(\theta,t) - e\psi_b'(\theta,t)] \times [\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\phi+2)] -$$

$$\left(\frac{r}{R}\right)^2 \psi_b'(\theta,t)[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)] +$$

$$\frac{\cos\phi}{(R-r\cos\phi)}\,[r\xi_b'(\theta,t)\sin\phi + r(\eta_b'(\theta,t) - e\psi_b'(\theta,t))\cos\phi + r^2\psi_b'(\theta,t)]\Bigg] +$$

$$\frac{1}{(R-r\cos\ )^2}\Bigg[(R-r\cos\phi)\sin\phi\cos\phi\left[\frac{1}{(R-r)}\,\zeta(\theta,t) - \left[\frac{1}{(R-r)} -\right.\right.$$

$$\left.\frac{1}{(R-r\cos\phi)}\right]\xi_b'(\theta,t) - \left[\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\ +2)\right]\left(\frac{r}{R}\right)^2 \times$$

$$[\eta_b'(\theta,t) - e\psi_b'(\theta,t)] - \left(\frac{r}{R}\right)^2 \psi_b'(\theta,t)[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)]\Bigg] +$$

$$\sin\phi\left[-(\xi_b'+\xi_s')\cos\phi + (\eta_b'+\eta_s')\sin\phi + (\psi_b'+\psi_s')n(\phi)]\right]\Bigg]^2 \qquad (3\text{-}45)$$

Taking the limit as $R\to\infty\left(\frac{1}{R}\frac{\partial}{\partial\theta}(\ldots) = \frac{\partial}{\partial z}(\ldots)\right)$ in (3-45) yields

$$\lim_{R\to\infty} U = \frac{Eh}{2(1-\nu^2)}\left[\left[\bar{\zeta}'(z,t) - \lim_{R\to\infty} R\left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right]\frac{1}{R^2}\,\xi_b'' -\right.\right.$$

$$\left. r\sin\phi[\bar{\eta}_b''(z,t) - e\bar{\psi}_b''(z,t)] - r^2\bar{\psi}_b''(z,t)\right]^2 +$$

$$\left(\frac{1-\nu}{2}\right)[\bar{\xi}_s'(z,t)\sin\phi + \bar{\eta}_s'(z,t)\cos\phi - \bar{\psi}_s'(z,t)h(\phi)]^2\Big] +$$

$$\frac{1}{2} G \frac{h^3}{3} (\bar{\psi}_b'+\bar{\psi}_s')^2 \qquad (3\text{-}46)$$

Taking a binomial series expansion of the unevaluated limit in (3-46) gives

$$R\left[\frac{1}{(1-a)} - \frac{1}{(1 - a\cos\ )}\right] = R[1+a+a^2+\ldots-1-a\cos\phi-a^2\cos^2\phi-\ldots]$$

$$= R[a(1-\cos\phi) + a^2(1-\cos^2\phi) + \ldots]$$

$$= \{r(1-\cos\phi) + \frac{r^2}{R}(1-\cos^2\phi) + \ldots\} \qquad (3\text{-}47)$$

Taking the limit as $R\to\infty$ in (3-47) gives

$$R\left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right] = r(1-\cos\phi) \qquad (3\text{-}48)$$

Substituting (3-48) into (3-46) gives the expression

$$\lim_{R\to\infty} U(\theta,\phi,t) = \bar{U}(z,\phi,t) = \frac{Eh}{2(1-\nu^2)}\Big[[\bar{\zeta}'(z,t) -$$

$$r(1-\cos\phi)\bar{\xi}_b''(z,t) - \bar{\eta}_b''(z,t)r\sin\phi + r\bar{\psi}_b''(z,t)[e\sin\phi-r\phi]\Big]^2 +$$

$$\left(\frac{1-\nu}{2}\right)\left[\bar{\xi}_s'(z,t)\sin\phi + \bar{\eta}_s'(z,t)\cos\phi - \bar{\psi}_s'(z,t)h(\phi)\right]^2 +$$

$$\frac{1}{2} G \frac{h^3}{3} [\bar{\psi}_b(z,t) + \bar{\psi}_s'(z,t)]^2 \tag{3-49}$$

This expression, subject to appropriate nomenclature changes, agrees with Tso's strain energy density function for the straight beam [37].

Using the strain energy function of the rigid cross-section displacements (3-45) in (3-30) gives the strain energy expression (3-32) where the integral coefficients are defined in Appendix B along with the "exact expressions" used in the Simpson's rule integration of the previous section.

Some of these integrals presented evaluation difficulties because they involve the difference of two numbers of almost equal magnitude. Consequently the computer-produced values could be several hundred per cent off. All of these troublesome integrals resulted from the integration of various functions of the R.H.S. of (3-22). One such integral is I_2 where

$$I_2 = hrR^2 \int_{-\alpha}^{\alpha} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\bar{\phi})}\right]^2 (1-a\cos\bar{\phi})d\bar{\phi}$$

$$= hrR^2 \left[\frac{1}{(1-a)^2} \int_{-\alpha}^{\alpha} (1-a\cos\bar{\phi})d\bar{\phi} - \frac{2}{(1-a)} \int_{-\alpha}^{\alpha} d\bar{\phi} + \int_{-\alpha}^{\alpha} \frac{d\bar{\phi}}{(1-a\cos\bar{\phi})}\right] \tag{3-50}$$

If the first two integrals on the R.H.S. of (3-50) are evaluated exactly and the third integral is evaluated using a binomial series expansion truncated after five terms, the value of I_2 is calculated as negative for some values of a. This is impossible since the integrand of (3-50) is always positive. This third integral can be evaluated exactly and its value is

$$\int_{-\alpha}^{\alpha} \frac{d\bar{\phi}}{(1-a\cos\bar{\phi})} = \frac{2}{\sqrt{1-a^2}} \tan^{-1}\left[\frac{\sqrt{1-a^2}\,\tan(\bar{\phi}/2)}{(1-a)}\right]\Bigg|_{-\alpha}^{\alpha} \tag{3-51}$$

but this is inconsistent with the treatment of other integrals within the system. Another approximate solution that simultaneously takes care of the problem of what happens to I_2 when the limit as $R \to \infty$ is taken is obtained by expanding the integrand of (3-50) in a five-term binomial series expansion, by cancelling terms of magnitude equal and opposite sign, and by integrating the resulting expression. This approximate answer for (3-50) is

$$I_2 \approx 2hr^3\Bigg[\left[\frac{3}{2}\alpha - 2\sin\alpha + \frac{1}{2}\sin\alpha\cos\alpha\right] + a\left[2\alpha + \frac{1}{3}\sin\alpha(\cos^2\alpha - 7)\right] + a^2\left[\frac{27}{8}\alpha + \frac{1}{4}\sin\alpha\cos^3\alpha + \frac{3}{8}\sin\alpha\cos\alpha - 4\sin\alpha\right]\Bigg] \tag{3-52}$$

Comparisons of the various solutions for I_2 are given in Table 2 for values of the radii ratio varying from zero to two-fifths.

Table 2. Comparisons of the Various Solutions of I_2 for Radii Ratios from 0.0 to 0.4

a	A	B	C
0.40	1.101	1.118	1.255
0.39	0.959	0.974	1.172
0.38	0.827	0.840	1.092
0.37	0.705	0.716	1.017
0.36	0.591	0.600	0.945
0.35	0.485	0.493	0.877
0.34	0.386	0.393	0.813
0.33	0.294	0.300	0.752
0.32	0.209	0.214	0.694
0.31	0.130	0.134	0.639
0.30	0.056	0.059	0.587
0.29	-0.013	-0.010	0.539
0.28	-0.076	-0.074	0.493
0.27	-0.135	-0.133	0.450
0.26	-0.190	-0.188	0.409
0.25	-0.241	-0.239	0.371
0.24	-0.288	-0.286	0.336
0.23	-0.331	-0.330	0.303
0.22	-0.371	-0.370	0.272
0.21	-0.408	-0.407	0.243
0.20	-0.442	-0.441	0.216
0.19	-0.473	-0.473	0.191
0.18	-0.501	-0.501	0.169
0.17	-0.528	-0.528	0.148
0.16	-0.552	-0.552	0.128
0.15	-0.573	-0.573	0.111
0.14	-0.593	-0.593	0.094
0.13	-0.611	-0.611	0.080
0.12	-0.627	-0.627	0.067
0.11	-0.641	-0.641	0.055
0.10	-0.654	-0.654	0.045
0.09	-0.665	-0.665	0.036
0.08	-0.674	-0.674	0.028
0.07	-0.682	-0.682	0.021
0.06	-0.689	-0.689	0.015
0.05	-0.695	-0.695	0.010
0.04	-0.700	-0.700	0.006
0.03	-0.703	-0.703	0.004
0.02	-0.705	-0.705	0.002
0.01	-0.707	-0.707	0.001
0.00	-0.707	-0.707	0.000

Column A represents the solution involving the exact integration for the first two integrals in (3-50) and the five-term expansion of the third; Column B represents the exact integration of I_2 and incorporates the results of (3-51); Column C is the solution resulting from the integration of the five-term binomial series expansion of the integrand given in (3-52). Columns A and B are seen to be in very good agreement for values of the radii ratio as large as three-tenths, and essentially, they are a comparison between the exact integration (3-51) and the approximate integration of the integral in (3-51) obtained through integration of the truncated binomial expansion of the integrand. Columns A and B are, however, incorrect since they are negative for radii ratio of 0.29 and less. This is attributed to the character of I_2 which is the difference between numbers of similar magnitude. Other integrals that require the same type examination if their numerical values are to mean anything are I_5, I_{10}, I_{19}, I_{21}, and I_{33}. These integrals were integrated by expanding the integrands into binomial series and truncating these series after 3 + n terms where n is the degree of the term

$$\left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right]^n .$$

That is, for n = 1, four terms were included and for n = 2, five terms were included.

Appendix B also contains numerical results for the various integrals with "typical" values of the input parameters given as

$$r = 10''$$

$$R = 50''$$

$$h = 0.05''$$

$$\phi_o = 10^\circ . \tag{3-53}$$

A comparison of the two techniques of solution is therefore readily available for these given parameters.

Kinetic Energy Expressions Using Simpson's Rule and Binomial Series Expansion Solutions

The same type of solutions used to evaluate the strain energy may be used to evaluate the kinetic energy density expression, T, defined as

$$T = \frac{1}{2}\int_{-\alpha}^{\alpha} \rho h[\dot{u}_1^2(\theta,\bar{\Phi},t) + \dot{u}_2^2(\theta,\bar{\Phi},t) + w(\theta,\bar{\Phi},t)]r\left(1-\frac{r}{R}\cos\bar{\Phi}\right)d\bar{\Phi} \tag{3-54}$$

where the rotary inertia of the shell about its midline has been neglected.

The velocities $\dot{u}_1,(\theta,\phi,t)$, $\dot{u}_2(\theta,\phi,t))$ and $\dot{w}(\theta,\phi,t)$ consistent with the deflections from the third section of this chapter are

$$\dot{u}_1(\theta,\phi,t) = [\dot{\xi}_b(\theta,t) + \dot{\xi}_s(\theta,t)]\sin\phi + [\dot{\eta}_b(\theta,t) + \dot{\eta}_s(\theta,t)]\cos\phi -$$

$$[\dot{\psi}_o(\theta,t) + \dot{\psi}_s(\theta,t)]h(\phi)$$

$$\dot{u}_2(\theta,\phi,t) = \frac{1 - a\cos\phi}{1 - a}\,\zeta(\theta,t) - \left[1 - \left(\frac{1-a\cos\phi}{1-a}\right)\right]\dot{\xi}_b'(\theta,t) -$$

$$\frac{2a}{(1+a)^2}\,(1-a\cos\phi)\left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan(\Phi/2)]} +\right.$$

$$\left.\frac{a}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\sqrt{\frac{1+a}{1-a}}\,\tan(\phi/2)\right]\right][\dot{\eta}_b'(\theta,t) - e\dot{\psi}_b'(\theta,t)] -$$

$$\frac{2a}{(1+a)^2}\,(1-a\cos\phi)\left[\frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan(\phi/2)]} +\right.$$

$$\left.\frac{1}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\sqrt{\frac{1+a}{1-a}}\,\tan(\phi/2)\right]\right]r\dot{\psi}_b'(\theta,t)$$

$$\dot{w}(\theta,\phi,t) = -[\dot{\xi}_b(\theta,t) + \dot{\xi}_s(\theta,t)]\cos\phi + [\dot{\eta}_b(\theta,t) +$$

$$\eta_s(\theta,t)]\sin\phi + [\dot{\psi}_b(\theta,t) + \dot{\psi}_s(\theta,t)]n(\phi)$$

where $n(\phi)$ and $h(\phi)$ are determined from (3-7) for a monosymmetric ring whose plane of symmetry coincides with the plane of the ring. That is

$$h(\phi) = e\cos\phi - r \tag{3-20}$$

$$n(\phi) = -e\sin\phi$$

The integral of the kinetic energy expression (3-54) is treated as the

$f(\phi)$ in the Simpson's one-third rule expression given in (3-31) via computer.

Velocities $\dot{u}_1(\theta,\phi,t)$, $\dot{u}_2(\theta,\phi,t)$ and $\dot{w}(\theta,\phi,t)$ consistent with the binomial series expansions of the fifth section of this chapter are

$$\dot{u}_1(\theta,\phi,t) = [\dot{\xi}_b(\theta,t) + \dot{\xi}_s(\theta,t)]\sin\phi + [\dot{\eta}_b(\theta,t) + \dot{\eta}_s(\theta,t)]\cos\phi - [\dot{\psi}_b(\theta,t) + \dot{\psi}_s(\theta,t)]h(\phi)$$

$$\dot{u}_2(\theta,\phi,t) = \frac{1 - a\cos\phi}{1 - a}\,\dot{\zeta}(\theta,t) - (R - r\cos\phi)\left\{\frac{1}{R}\,\dot{\xi}_b'(\theta,t) \times \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right] + \frac{r}{R^2}\left[\dot{\eta}_b'(\theta,t) + \dot{\eta}_s'(\theta,t)\right] \times \right.$$
$$[\sin\phi + a(\phi + \tfrac{1}{2}\sin 2\phi) + a^2\sin\phi(2 + \cos^2\phi)] +$$
$$\left.\left(\frac{r}{R}\right)^2 \dot{\psi}_b'(\theta,t)[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)]\right\} \qquad (3\text{-}56)$$

$$w(\theta,\phi,t) = -[\dot{\xi}_b(\theta,t) + \dot{\xi}_s(\theta,t)]\cos\phi + [\dot{\eta}_b(\theta,t) + \dot{\eta}_s(\theta,t)]\sin\phi + [\dot{\psi}_b(\theta,t) + \dot{\psi}_s(\theta,t)]\eta(\phi)$$

The kinetic energy density (3-54) for both the Simpson's Rule integration and the integral expansions is

$$T = \frac{1}{2}\Big[M_1[\dot{\xi}_b(\theta,t) + \dot{\xi}_s(\theta,t)]^2 + M_2[(\dot{\eta}_b+\dot{\eta}_s) - e(\dot{\psi}_b+\dot{\psi}_s)]^2 +$$

$$M_3 r^2[\dot{\psi}_6(\theta,t) + \dot{\psi}_s(\theta,t)]^2 + 2M_4 r(\dot{\psi}_b+\dot{\psi}_s)[(\dot{\eta}_b+\dot{\eta}_s) - e(\dot{\psi}_b+\dot{\psi}_s)] +$$

$$M_5\dot{\zeta}^2(\theta,t) + M_6 \frac{1}{R^2} \dot{\xi}_b'^2(\theta,t) + M_7 \frac{1}{R^2} (\dot{\eta}_b' - e\dot{\psi}_b')^2 + M_8\left(\frac{r}{R}\right)^2 \dot{\psi}_b'^2(\theta,t) -$$

$$M_9 \frac{2}{R} \dot{\zeta}(\theta,t)\dot{\xi}_b'(\theta,t) - M_{10} \frac{r}{R^2} \dot{\psi}_b'(\theta,t)[\dot{\eta}_b'(\theta,t) - e\dot{\psi}_b'(\theta,t)] +$$

$$M_{11}[\dot{\xi}_b(\theta,t) + \dot{\eta}_s(\theta,t)]^2 + M_{12}[(\dot{\eta}_b+\dot{\eta}_s) - e(\dot{\psi}_b+\dot{\psi}_s)]^2\Big] \qquad (3\text{-}57)$$

where M_1 through M_{12} are defined in Appendix B as exact expressions for Simpson's Rule and approximate expressions for the binomial series expansions.

In-Plane and Out-of-Plane Equations of Motion

Forming the change in energy for the system gives the equation

$$-\int_{t_1}^{t_2}\int_0^{\theta_o} (T-V)Rd\theta dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^{\theta_o} \Big\{-[M_1(\dot{\xi}_b+\dot{\xi}_s)^2 + M_2[(\dot{\eta}_b+\dot{\eta}_s) -$$

$$e(\dot{\psi}_b+\dot{\psi}_s)]^2 + M_3 r^2(\dot{\psi}_b+\dot{\psi}_s)^2 + M_4 2r(\dot{\psi}_b+\dot{\psi}_s)[(\dot{\eta}_b+\dot{\eta}_s) - e(\dot{\psi}_b+\dot{\psi}_s)] +$$

$$M_5\dot{\zeta}^2(\theta,t) + M_6 \frac{1}{R^2} \dot{\xi}_b'^2(\theta,t) + M_7 \frac{1}{R^2} (\dot{\eta}_b'-e\dot{\psi}_b')^2 + M_8\left(\frac{r}{R}\right)^2 \dot{\psi}_b'^2(\theta,t) -$$

$$M_9 \frac{2}{R} \dot{\zeta}\dot{\xi}_b' + 2M_{10} \frac{r}{R^2} \dot{\psi}_b'(\theta,t) \times (\dot{\eta}_b'-e\dot{\psi}_b') + M_{11}(\dot{\xi}_b+\dot{\xi}_s)^2 + M_{12}[(\dot{\eta}_b+\dot{\eta}_s) -$$

$$e(\dot{\psi}_b+\dot{\psi}_s)]^2\Big] + \frac{E}{(1-\nu^2)}\Big[I_1\frac{1}{R^2}\zeta'^2(\theta,t) + I_2\frac{1}{R^4}\xi_b''^2(\theta,t) +$$

$$I_3\frac{1}{R^4}(\eta_b''-e\psi_b'')^2 + I_4\left(\frac{r}{R^2}\psi_b''\right)^2 + I_5\frac{2}{R}\zeta'(\theta,t)\frac{1}{R^2}\xi_b''(\theta,t) +$$

$$I_6\frac{2r}{R^4}\psi_b''(\theta,t)(\eta_b''-e\psi_b'') + I_7(\xi_b+\xi_s)^2 + I_8r^2(\psi_b+\psi_s)^2 +$$

$$I_9\frac{2}{R}\zeta'(\theta,t)(\xi_b+\xi_s) + I_{10}\frac{2}{R^2}\xi_b''(\xi_b+\xi_s) - I_{11}\frac{2r}{R^2}(\psi_b+\psi_s)(\eta_b''-e\psi_b'') -$$

$$I_{12}2\left(\frac{r}{R}\right)^2\psi_b''(\psi_b+\psi_s)] + G\Big[I_{13}\left(\frac{1}{R}\xi_s'\right)^2 + I_{14}\frac{1}{R^2}(\eta_s'-e\psi_s')^2 + I_{15}\left(\frac{r}{R}\psi_s'\right)^2 +$$

$$I_{16}\left(2\frac{r}{R^2}\right)\psi_s'(\eta_s'-e\psi_s')] + \frac{G}{3}\Big[I_{17}\frac{r}{R}^2(\psi_b'+\psi_s')^2 + I_{18}\zeta^2(\theta,t) +$$

$$I_{19}\frac{1}{R^2}\xi_b'^2(\theta,t) + I_{20}\frac{1}{R^2}(\eta_b'-e\psi_b')^2 - I_{21}\frac{2}{R}\zeta(\theta,t)\times\xi_b'(\theta,t) +$$

$$I_{22}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) + I_{23}\frac{2r}{R^2}\psi_b'(\eta_b'-e\psi_b') + I_{24}\frac{1}{R^2}\xi_b'^2 + I_{25}\frac{1}{R^2}(\eta_b'-e\psi_b')^2 +$$

$$I_{26}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) + I_{27}\left(\frac{2r}{R^2}\right)\psi_b'(\theta,t)(\eta_b'-e\psi_b') + I_{28}\left(\frac{2r}{R^2}\right)(\psi_b'+\psi_s')(\eta_b'-e\psi_b') +$$

$$I_{29}\left(\frac{2r^2}{R^2}\right)\psi_b'(\psi_b'+\psi_s') + I_{30}\left(\frac{2r}{R^2}\right)(\psi_b'+\psi_s')(\eta_b'-e\psi_b') + I_{31}2\left(\frac{r}{R}\right)^2\psi_b'(\psi_b'+\psi_s') -$$

$$I_{32}\frac{2}{R}\zeta(\theta,t)\xi_b'(\theta,t) + I_{33}\frac{2}{R^2}\xi_b'^2(\theta,t) + I_{34}\frac{2}{R^2}(\eta_b'-e\psi_b')^2 +$$

$$I_{35}\,\frac{2r}{R^2}\,\psi_b'(\theta,t)\times(\eta_b'-e\psi_b') + I_{36}\left(\frac{2r}{R^2}\right)\psi_b'(\eta_b'-e\psi_b') + I_{37}2\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) +$$

$$I_{18}\zeta^2(\theta,t) + I_{19}\,\frac{1}{R^2}\,\xi_b'^2(\theta,t) + I_{20}\,\frac{1}{R^2}\,(\eta_b'-\psi_b')^2 + I_{22}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) -$$

$$I_{21}\,\frac{2}{R}\,\zeta(\theta,t)\xi_b'(\theta,t) + I_{23}\left(\frac{2r}{R^2}\right)\psi_b'(\eta_b'-e\psi_b') + I_{24}\,\frac{1}{R^2}\,(\xi_b'+\xi_s')^2 +$$

$$I_{38}\,\frac{1}{R^2}\,[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')]^2 - I_{32}\,\frac{2}{R}\,\zeta(\theta,t)(\xi_b'+\xi_s') +$$

$$I_{33}\,\frac{2}{R^2}\,\xi_b'(\theta,t)(\xi_b'+\xi_s') - I_{39}\,\frac{2}{R^2}\,(\eta_b'-e\psi_b')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] -$$

$$I_{40}\left(\frac{2r}{R^2}\right)\psi_b'(\theta,t)\times[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] + 2\left[-I_{41}\left(\frac{r}{R^2}\right)(\psi_b'+\psi_s')(\eta_b'-e\psi_b') -\right.$$

$$I_{42}\left(\frac{r}{R}\right)^2\psi_b'(\psi_b'+\psi_s') + I_{43}\left(\frac{r}{R^2}\right)(\psi_b'+\psi_s')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] - I_{18}\zeta^2(\theta,t) +$$

$$I_{21}\,\frac{1}{R}\,\xi_b'(\theta,t)\zeta(\theta,t) + I_{32}\,\frac{1}{R}(\xi_b'+\xi_s')\zeta(\theta,t) + I_{21}\,\frac{1}{R}\,\zeta(\theta,t)\xi_b'(\theta,t) -$$

$$I_{19}\,\frac{1}{R^2}\,\xi_b'^2(\theta,t) - I_{33}\,\frac{1}{R^2}\,(\xi_b'+\xi_s')\xi_b'(\theta,t) - I_{20}\,\frac{1}{R^2}\,(\eta_b'-e\psi_b')^2 -$$

$$I_{23}\,\frac{r}{R^2}\,\psi_b'(\eta_b'-e\psi_b') + I_{39}\,\frac{1}{R^2}\,(\eta_b'-e\psi_b')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] -$$

$$I_{23}\,\frac{r}{R^2}\,\psi_b'(\theta,t)(\eta_b'-e\psi_b') - I_{22}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) + I_{40}\left(\frac{r}{R^2}\right)\psi_b'[(\eta_b'+\eta_s') -$$

$$e(\psi_b'+\psi_s')] + I_{32}\frac{1}{R}\zeta(\theta,t)\xi_b'(\theta,t) - I_{33}\frac{1}{R^2}\xi_b'^2(\theta,t) - I_{34}\frac{1}{R^2}(\eta_b'-e\psi_b')^2 -$$

$$I_{36}\frac{r}{R^2}\psi_b'(\theta,t)(\eta_b'-e\psi_b') - I_{35}\frac{r}{R^2}\psi_b'(\eta_b'-e\psi_b') - I_{37}\left(\frac{r}{R}\right)^2\psi_b'^2(\theta,t) -$$

$$I_{24}\frac{1}{R^2}\xi_b'(\theta,t)(\xi_b'+\xi_s') + I_{24}\frac{1}{R^2}(\eta_b'-e\psi_b')[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')] +$$

$$I_{44}\left(\frac{r}{R^2}\right)\psi_b'[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')]\Bigg]\Bigg]R\,d\theta\,dt \qquad (3\text{-}58)$$

where all coefficients are defined in Appendix B.

The final step in obtaining the equations of motion involves performing the variational procedure necessary to satisfy (3-1). This can be carried out expediently using the Euler-Lagrange equations [39] which are for the number of dependent and independent variables as follows:

$$\frac{\partial F}{\partial \zeta} - \frac{\partial}{\partial \theta}\left(\frac{\partial F}{\partial \zeta'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\zeta}}\right) = 0$$

$$\frac{\partial F}{\partial \xi_b} - \frac{\partial}{\partial \theta}\left(\frac{\partial F}{\partial \xi_b'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\xi}_b}\right) + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial F}{\partial \xi_b''}\right) + \frac{\partial^2}{\partial \theta \partial t}\left(\frac{\partial F}{\partial \dot{\xi}_b'}\right) = 0$$

$$\frac{\partial F}{\partial \xi_s} - \frac{\partial}{\partial \theta}\left(\frac{\partial F}{\partial \xi_s'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\xi}_s}\right) = 0 \qquad (3\text{-}59)$$

$$- \frac{\partial}{\partial \theta}\left(\frac{\partial F}{\partial \eta_b'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\eta}_b}\right) + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial F}{\partial \eta_b''}\right) + \frac{\partial^2}{\partial \theta \partial t}\left(\frac{\partial F}{\partial \dot{\eta}_b'}\right) = 0$$

$$-\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial \eta_s'}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\eta}_s}\right) = 0$$

$$\frac{\partial F}{\partial \psi_b}-\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial \psi_b'}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\psi}_b}\right)+\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial F}{\partial \psi_b''}\right)+\frac{\partial^2}{\partial\theta\partial t}\left(\frac{\partial F}{\partial \dot{\psi}_b'}\right) = 0$$

$$\frac{\partial F}{\partial \psi_s}-\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial \psi_s'}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\psi}_s}\right) = 0$$

where

$$F = T(\theta,t) - U(\theta,t)$$

and

$$F = F(\zeta,\zeta',\dot{\zeta},\zeta_b,\xi_b',\dot{\xi}_b,\xi_b'',\dot{\xi}_b';\xi_s,\xi_s',\dot{\xi}_s;\eta_b',\dot{\eta}_b,\dot{\eta}_b'',\dot{\eta}_b';$$

$$\eta_s',\dot{\eta}_s,\psi_b,\psi_b',\dot{\psi}_b,\psi_b'',\dot{\psi}_b',\psi_s,\psi_s',\dot{\psi}_s;\theta,t)$$

Carrying out the operations of (3-59), the resulting three in-plane equations of motion and four out-of-plane equations of motion after considerable manipulation are, respectively,

$$-M_5\ddot{\zeta} + E^*I_1\frac{1}{R^2}\zeta'' + M_9\frac{1}{R}\ddot{\xi}_b' + E^*I_5\frac{1}{R^3}\xi_b''' +$$

$$E^*I_9\frac{1}{R}\xi_b' + E^*I_9\frac{1}{R}\xi_s' = 0 \qquad (3\text{-}60)$$

$$M_9 \frac{1}{R} \ddot{\zeta}' + E^*I_5 \frac{1}{R^3} \zeta''' + E^*I_9 \frac{1}{R} \zeta' + M_{13}\ddot{\xi}_b - M_6 \frac{1}{R^2} \ddot{\xi}_b'' +$$

$$E^*I_2 \frac{1}{R^4} \xi_b'''' + 2E^*I_{10} \frac{1}{R^2} \xi_b'' + E^*I_7\xi_b + M_{13}\ddot{\xi}_s +$$

$$E^*I_{10} \frac{1}{R^2} \xi_s'' + E^*I_7\xi_s = 0 \qquad (3\text{-}61)$$

$$E^*I_9 \frac{1}{R} \zeta' + M_{13}\ddot{\xi}_b + E^*I_{10} \frac{1}{R^2} \xi_b'' + E^*I_7\xi_b + M_{13}\ddot{\xi}_s +$$

$$E^*I_7\xi_s - GI_{52} \frac{1}{R^2} \xi_s'' = 0 \qquad (3\text{-}62)$$

$$M_{14}\ddot{\eta}_b - M_7 \frac{1}{R^2} \ddot{\eta}_b'' + E^*I_3 \frac{1}{R^2} \eta_b'''' - \frac{1}{3} GI_{53} \frac{1}{R^2} \eta_b'' + M_{14}\ddot{\eta}_s - \frac{1}{3} GI_{47} \frac{1}{R^2} \eta_s'' +$$

$$M_{16}r\ddot{\psi}_b + M_{15} \frac{r}{R^2} \ddot{\psi}_b'' - \frac{1}{3} GI_{54} \frac{r}{R^2} \psi_b'' + E^*I_{55} \frac{r}{R^4} \psi_b'''' -$$

$$E^*I_{11} \frac{r}{R^2} \psi_b'' + M_{16}r\ddot{\psi}_s - E^*I_{11} \frac{r}{R^2} \psi_s'' - \frac{1}{3} GI_{56} \frac{r}{R^2} \psi_s'' = 0 \qquad (3\text{-}63)$$

$$M_{14}\ddot{\eta}_b - \frac{1}{3} GI_{47} \frac{1}{R^2} \eta_b'' + M_{14}\eta_s - GI_{57} \frac{1}{R^2} \eta_s'' + M_{16}r\ddot{\psi}_b - \frac{1}{3} GI_{51} \frac{r}{R^2} \psi_b'' +$$

$$M_{16}r\ddot{\psi}_s - GI_{58} \frac{r}{R^2} \psi_s'' = 0 \qquad (3\text{-}64)$$

$$M_{16}\ddot{\eta}_b + M_{15} \frac{1}{R^2} \ddot{\eta}_b'' - \frac{1}{3} GI_{54} \frac{1}{R^2} \eta_b'' + E^*I_{55} \frac{1}{R^4} \eta_b'''' - E^*I_{11} \frac{1}{R^2} \eta_b'' + M_{16}\ddot{\eta}_s -$$

$$\frac{1}{3} GI_{51} \frac{1}{R^2} \eta_s'' + M_{17}r\ddot{\psi}_b - M_{18} \frac{r}{R^2} \ddot{\psi}_b'' + E^*I_{59} \frac{r}{R^2} \psi_b'''' -$$

$$2E^*I_{60}\,\frac{r}{R^2}\,\psi_b'' + E^*I_8 r\psi_b - \frac{1}{3}\,GI_{61}\,\frac{r}{R^2}\,\psi_b'' + M_{17}r\ddot{\psi}_s - E^*I_{60}\,\frac{r}{R^2}\,\psi_s'' +$$

$$E^*I_8 r\psi_s - \frac{1}{3}\,GI_{63}\,\frac{r}{R^2}\,\psi_s'' = 0 \qquad (3\text{-}65)$$

$$M_{16}\ddot{\eta}_b - E^*I_{11}\,\frac{1}{R^2}\,\eta_b'' - \frac{1}{3}\,GI_{56}\,\frac{1}{R^2}\,\eta_b'' + M_{16}\ddot{\eta}_s - GI_{58}\,\frac{1}{R^2}\,\eta_s'' + M_{17}r\ddot{\psi}_b -$$

$$\frac{G}{3}\,I_{63}\,\frac{r}{R^2}\,\psi_b'' - E^*I_{60}\,\frac{r}{R^2}\,\psi_b'' + E^*I_8 r\psi_b + M_{17}r\ddot{\psi}_s + E^*I_8 r\psi_s -$$

$$GI_{62}\,\frac{r}{R^2}\,\psi_s'' = 0 \qquad (3\text{-}66)$$

where

$$I_{45} = I_{24} + I_{25}$$

$$I_{46} = I_{28} + I_{30} - I_{41} + I_{43}$$

$$I_{47} = I_{24} + I_{38}$$

$$I_{48} = I_{17} + I_{29} + I_{31} - I_{42} + \left(\frac{e}{r}\right)[I_{41} - I_{43} - I_{28} - I_{30}]$$

$$I_{49} = I_{27} + I_{28} + I_{30} - I_{41} - \left(\frac{e}{r}\right)[I_{24} + I_{25}]$$

$$I_{50} = I_{26} + I_{29} + I_{31} - I_{42} - \left(\frac{e}{r}\right)[I_{27} + I_{44}]$$

$$I_{51} = I_{43} + I_{44} - \left(\frac{e}{r}\right) [I_{24} + I_{25}]$$

$$I_{52} = I_{13} + \frac{1}{3} I_{24}$$

$$I_{53} = I_{45} + I_{47}$$

$$I_{54} = I_{27} + I_{46} - \left(\frac{e}{r}\right) [I_{45} + I_{47}]$$

$$I_{55} = I_{6} - \left(\frac{e}{r}\right) I_{3}$$

$$I_{56} = I_{46} - \left(\frac{e}{r}\right) I_{47}$$

$$I_{57} = I_{14} + \frac{1}{3} I_{38}$$

$$I_{58} = I_{16} + \frac{1}{3} I_{43} - \left(\frac{e}{r}\right) [I_{14} + \frac{1}{3} I_{38}]$$

$$I_{59} = I_{4} - 2\left(\frac{e}{r}\right) I_{6} + \left(\frac{e}{r}\right)^{2} I_{3}$$

$$I_{60} = I_{12} - \left(\frac{e}{r}\right) I_{11}$$

$$I_{61} = I_{48} + I_{50} - \left(\frac{e}{r}\right) [I_{49} + I_{51}]$$

$$I_{62} = I_{15} + \frac{1}{3} I_{17} - 2 \left(\frac{e}{r}\right) I_{16} + \left(\frac{e}{r}\right)^{2} [I_{14} + \frac{1}{3} I_{38}]$$

$$I_{63} = I_{48} - \left(\frac{e}{r}\right) I_{51}$$

$$M_{13} = M_1 + M_{11}$$

$$M_{14} = M_2 + M_{12}$$

$$M_{15} = -M_{10} + \left(\frac{e}{r}\right)M_7$$

$$M_{16} = M_4 - \left(\frac{e}{r}\right)M_{21}$$

$$M_{17} = M_3 - 2.0\left(\frac{e}{r}\right)M_4 + \left(\frac{e}{r}\right)^2 M_{21}$$

$$M_{18} = M_8 - 2.0\left(\frac{e}{r}\right)M_{10} + \left(\frac{e}{r}\right)^2 M_7$$

The boundary conditions associated with the equations of motion are a by-product of the variational derivation. These relations are for the seven equations of motion, respectively,

$$\int_{t_1}^{t_2} \left(\frac{\partial F}{\partial \zeta'}\right)\delta\zeta\Big|_0^{\theta_o} dt = 0 \tag{3-67}$$

$$\frac{E^*}{R}\left[I_1 \frac{1}{R}\zeta'(\theta,t) + I_5 \frac{1}{R^2}\xi_b''(\theta,t) + I_9(\xi_b+\xi_s)\right]\delta\zeta\Big|_0^{\theta_o} = 0$$

$$\int_{t_1}^{t_2}\left[\frac{\partial F}{\partial \xi_b'}\delta\xi_b + \frac{\partial F}{\partial \xi_b''}\delta\xi_b' - \frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial \xi_b''}\right)\delta\xi_b - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\xi}_b'}\right)\delta\xi_b\right]_0^{\theta_o} dt = 0 \tag{3-68}$$

$$\frac{1}{R}\left[\frac{G}{3}(I_{21}-I_{33})\zeta(\theta,t) - E^*\left(I_2\frac{1}{R^3}\xi_\beta''' + I_{10}\frac{1}{R}(\xi_b'+\xi_s')\right)\right]\delta\xi_b\Big|_0^{\theta_o} +$$

$$\frac{1}{R}\left[E^*\left(I_2 \frac{1}{R^2} \xi_b'' + I_5 \frac{1}{R} \zeta' + I_{10}(\xi_b+\xi_s)\right)\right]\delta\xi_b'\bigg|_0^{\theta_o} + E^* I_5 \frac{1}{R^2} \zeta''\delta\xi_b\bigg|_0^{\theta_o} +$$

$$\frac{1}{R}\left[M_b \frac{1}{R} \ddot{\xi}_b' - \dot{M}_g\ddot{\zeta}\right]\delta\xi_b\bigg|_0^{\theta_o} = 0$$

$$\int_{t_1}^{t_2} \frac{\partial F}{\partial \xi_s'} \delta\xi_s\bigg|_0^{\theta_o} dt = 0 \tag{3-69}$$

$$\frac{G}{R}\left[I_{13} \frac{1}{R} \xi_s' + I_{24} \frac{1}{R} \xi_s'\right]\delta\xi_s\bigg|_0^{\theta_o} = 0$$

$$\int_{t_1}^{t_2} \left[\left[\frac{\partial F}{\partial \eta_b'} - \frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial \eta_b''}\right)\right]\delta\eta_b + \frac{\partial F}{\partial \eta_b''} \delta\eta_b'\right]_0^{\theta_o} dt = 0 \tag{3-70}$$

$$\left[\frac{1}{R} \frac{G}{3}\left[I_{45} \frac{1}{R}(\eta_b'-e\psi_b') + I_{27} \frac{r}{R} \psi_b' + I_{46} \frac{r}{R}(\psi_b'+\psi_s') +\right.\right.$$

$$\left.I_{47} \frac{1}{R} \left((\eta_b'+\eta_s') - e(\psi_b'+\psi_s')\right)\right] - \frac{E^*}{R}\left[I_3 \frac{1}{R^3} (\eta_b'''+e\,\psi_b''') -\right.$$

$$\left.\left.I_6 \frac{r}{R^3} \psi_b''' - I_{11} \frac{r}{R} (\psi_b'+\psi_s')\right)\right]\delta\eta_b\bigg|_0^{\theta_o} +$$

$$\left[\frac{E^*}{R}\left[I_3 \frac{1}{R^2}(\eta_b''-e\psi_b'') - I_6 \frac{r}{R^2} \psi_b'' - I_{11} r(\psi_b+\psi_s)\right] \quad \frac{1}{R}\delta\eta_b'\right|_0^{\theta_o} = 0$$

$$\int_{t_1}^{t_2} \frac{\partial F}{\partial \eta_s'} \delta\eta_s\bigg|_0^{\theta_o} dt = 0 \tag{3-71}$$

$$\left[\frac{G}{R}\left[I_{14}\,\frac{1}{R}\,(\eta_s'-e\psi_s') + I_{16}\,\frac{r}{R}\,\psi_s'\right] + \frac{G}{3R}\left[I_{38}\,\frac{1}{R}\,\times\right.\right.$$

$$\left.\left\{(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')\right\} + I_{24}\,\frac{1}{R}\,(\eta_b'-e\psi_b') + I_{44}\,\frac{r}{R}\,\psi_b' +\right.$$

$$\left.\left.I_{43}\,\frac{r}{R}\,(\psi_b'+\psi_s')\right]\right]\delta\eta_s\Big|_0^{\theta_o} = 0$$

$$\int_{t_1}^{t_2}\left[\left[\frac{\partial F}{\partial\psi_b'} - \frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial\psi_b''}\right)\right]r\delta\psi_b + \frac{\partial F}{\partial\psi_b''}\left(\frac{r}{R}\right)\delta\psi_b'\right]_0^{\theta_o} dt = 0 \tag{3-72}$$

$$\left[\frac{G}{3}\left[I_{48}\,\frac{r}{R^2}\,(\psi_b'+\psi_s') + I_{49}\,\frac{1}{R}\,(\eta_b'-e\psi_b') + I_{50}\,\frac{r}{R}\,\psi_b' +\right.\right.$$

$$\left.I_{51}\,\frac{1}{R}\left[(\eta_b'+\eta_s') - e(\psi_b'+\psi_s')\right]\right] - E^*\left[\left\{I_4 + \left(\frac{e}{r}\right)I_6\right\}\frac{r}{R^3}\,\psi_b''' -\right.$$

$$\left.\left.\left\{I_6 + \left(\frac{e}{r}\right)I_3\right\}\frac{1}{R^3}(\eta_b''' - e\psi_b''') - I_{60}\,\frac{r}{R}\,(\psi_b'+\psi_s')\right]\right]\left(\frac{r}{R}\right)\delta\psi_b\Big|_0^{\theta_o} +$$

$$E^*\left\{I_4 + \left(\frac{e}{r}\right)I_6\right\}\frac{r}{R^2}\,\psi_b'' - I_{55}\,\frac{1}{R^2}\,(\eta_b''-e\psi_b'') - I_{60}r(\psi_b+\psi_s)\quad\frac{r}{R^2}\,\delta\psi_b'\Big|_0^{\theta_o} = 0$$

$$\int_{t_1}^{t_2}\frac{\partial F}{\partial\psi_s'}\,\delta\psi_s\Big|_0^{\theta_o} dt = 0 \tag{3-73}$$

$$G\left[\left\{I_{16} - \left(\frac{e}{r}\right) I_{14}\right\}\frac{1}{R}\,(\eta_s' - e\psi_s') + \left\{I_{15} - \left(\frac{e}{r}\right) I_{16}\right\}\frac{r}{R}\,\psi_s'(\theta,t) + \right.$$

$$\frac{1}{3}\left\{I_{17} - \left(\frac{e}{r}\right) I_{43}\right\}\frac{r}{R}(\psi_b' + \psi_s') + I_{100}\,\frac{1}{R}(\eta_b' - e\psi_b') +$$

$$\left. I_{101}\left(\frac{r}{R}\right)\psi_b' + \left\{I_{43} + \left(\frac{e}{r}\right) I_{38}\right\}\frac{1}{R}[(\eta_b' + \eta_s') - e(\psi_b' + \psi_s')]\right]\frac{r}{R}\,\delta\psi_s\Big|_0^{\theta_o} = 0$$

The resulting equations of motion are a set of seven coupled partial differential equations with independent variables θ and t. There are inherent static couplings due to the monosymmetric cross-section. The equations thus constitute a theory for a thin-walled curved beam with a cut-circular cross-section analogous to the Timoshenko beam theory.

Simplified Equations of Motion

Equations of motion for free oscillations of a ring analogous to equations in the literature can be deduced from (3-60) through (3-66) by elimination of certain terms.

Omitting shear and rotary inertia effects can be accomplished by equating ξ_s, η_s, and ψ_s to zero, by eliminating the shear equations (3-62), (3-64), and (3-66), and by neglecting inertia terms involving spatial derivatives. These simplified equations are for in-plane and out-of-plane free vibrations, respectively.

$$E^*I_1\,\frac{1}{R^2}\,\zeta'' - M_5\ddot{\zeta} + E^*I_5\,\frac{1}{R^3}\,\xi_b''' + E^*I_9\,\frac{1}{R}\,\xi_b' = 0 \qquad (3\text{-}74)$$

$$E^*I_5 \frac{1}{R^3} \zeta''' + E^*I_9 \frac{1}{R} \zeta' + E^*I_2 \frac{1}{R^4} \xi_b'''' +$$

$$2E^*I_{10} \frac{1}{R^2} \xi_b'' + E^*I_7 \xi_b + M_{20} \ddot{\xi}_b = 0 \tag{3-75}$$

$$E^*I_3 \frac{1}{R^4} \eta_b'''' - \frac{1}{3} GI_{53} \frac{1}{R^2} \eta_b'' + M_{14} \ddot{\eta}_b + E^*I_{55} \frac{1}{R^4} r\psi_b'''' - E^*I_{11} \frac{1}{R^2} r\psi_b'' -$$

$$\frac{1}{3} GI_{54} \frac{1}{R^2} r\psi_b'' + M_{16} r\ddot{\psi}_b = 0 \tag{3-76}$$

$$E^*I_{55} \frac{1}{R^4} \eta_b'''' - E^*I_{11} \frac{1}{R^2} \eta_b'' - \frac{1}{3} GI_{54} \frac{1}{R^2} \eta_b'' + M_{16} \ddot{\eta}_b + E^*I_{59} \frac{r}{R^4} \psi_b'''' -$$

$$2E^*I_{60} \frac{r}{R^2} \psi_b'' + E^*I_8 r\psi_b - \frac{1}{3} GI_{61} \frac{r}{R^2} \psi_b'' + M_{17} r\ddot{\psi}_b = 0 \tag{3-77}$$

Comparisons between results of these equations and those of the conventional equations derived in Chapter II are examined in the following chapter.

If the radii ratio, r/R, is assumed to be very much less than one so that it may be neglected in comparison to one, then the equations of motion may be derived with coefficients of the differential terms composed of integrals whose integrands are functions of $x-a_x$ and $y-a_y$ as defined in (3-19). These equations are similar in form to (3-60) through (3-66). Taking the limit as $R\to\infty \left[\frac{1}{R} \frac{\partial}{\partial \theta} (\ldots) = \frac{\partial}{\partial z} (\ldots)\right]$ gives Tso's [37] equations of free oscillation for a straight beam with appropriate changes in nomenclature and signs. These equations are

$$E^*A\bar{\zeta}'' - \rho A\ddot{\bar{\zeta}} = 0$$

$$E^*I_{yy}\bar{\xi}_b'''' - \rho I_{yy}\ddot{\bar{\xi}}_b'' + \rho A(\ddot{\bar{\xi}}_b+\ddot{\bar{\xi}}_s) - \rho Aa_y(\ddot{\bar{\psi}}_b+\ddot{\bar{\psi}}_s) = 0$$

$$G(I_{ss}\bar{\xi}_s''+I_{sc}\bar{\eta}_s''-I_{hs}\bar{\psi}_s'') + \rho Aa_y(\ddot{\bar{\psi}}_b+\ddot{\bar{\psi}}_s) - \rho A(\ddot{\bar{\xi}}_b+\ddot{\bar{\xi}}_s) = 0$$

$$E^*I_{yy}\bar{\eta}_b''' - \rho I_{xx}\ddot{\bar{\eta}}_b'' + \rho A(\ddot{\bar{\eta}}_b+\ddot{\bar{\eta}}_s) + \rho Aa_x(\ddot{\bar{\psi}}_b+\ddot{\bar{\psi}}_s) = 0$$

$$G(I_{sc}\bar{\xi}_s''+I_{cc}\bar{\eta}_s''-I_{hc}\bar{\psi}_s'') - \rho A(\ddot{\bar{\eta}}_b+\ddot{\bar{\eta}}_s) - \rho Aa_x(\ddot{\bar{\psi}}_b+\ddot{\bar{\psi}}_s) = 0$$

$$E^*r^2I_{\omega\omega}\bar{\psi}_b'''' - \rho I_{\omega\omega}\ddot{\bar{\psi}}_b'' + \rho I_p(\ddot{\bar{\psi}}_b+\ddot{\bar{\psi}}_s) - \rho Aa_y(\ddot{\bar{\xi}}_b+\ddot{\bar{\xi}}_s) +$$

$$\rho Aa_x(\ddot{\bar{\eta}}_b+\ddot{\bar{\eta}}_s) - GI_d(\bar{\psi}_b''+\bar{\psi}_s'') = 0$$

$$G(I_{hs}\bar{\xi}_s''+I_{hc}\bar{\eta}_s''-I_{hh}\bar{\psi}_s'') + \rho I_p(\bar{\psi}_b+\bar{\psi}_s) - \rho Aa_y(\ddot{\bar{\xi}}_b+\ddot{\bar{\xi}}_s) +$$

$$\rho Aa_x(\ddot{\bar{\eta}}_b+\ddot{\bar{\eta}}_s) - GI_d(\bar{\psi}_b''+\bar{\psi}_s'') = 0 \qquad (3\text{-}78)$$

where

$$I_{yy} = \int_{-\alpha}^{\alpha} x^2(\phi)hrd\phi$$

$$A = \int_{-\alpha}^{\alpha} hrd\phi$$

$$I_{ss} = \int_{-\alpha}^{\alpha} \sin^2\bar{\phi} hrd\bar{\phi}$$

$$I_{sc} = \int_{-\alpha}^{\alpha} \sin\bar{\phi}\cos\bar{\phi} hrd\bar{\phi}$$

$$I_{hc} = \int_{-\alpha}^{\alpha} h(\bar{\phi})\cos\bar{\phi} hrd\bar{\phi}$$

$$I_{xx} = \int_{-\alpha}^{\alpha} y^2(\bar{\phi}) hrd\bar{\phi}$$

$$I_{cc} = \int_{-\alpha}^{\alpha} \cos^2\bar{\phi} hrd\bar{\phi}$$

$$I_{\omega\omega} = \int_{-\alpha}^{\alpha} \omega^2(\bar{\phi}) hrd\bar{\phi}$$

$$I_{p} = \int_{-\alpha}^{\alpha} [h^2(\bar{\phi}) + n^2(\bar{\phi})] hrd\bar{\phi}$$

$$I_{d} = r\frac{h^3}{3}\int_{-\alpha}^{\alpha} d\bar{\phi}$$

$$I_{hs} = \int_{-\alpha}^{\alpha} h(\bar{\phi})\sin\bar{\phi} hrd\bar{\phi}$$

$$I_{hh} = \int_{-\alpha}^{\alpha} h^2(\bar{\phi}) hrd\bar{\phi}$$

and

$$\omega(\phi) = \int_{0}^{\phi} h(\bar{\phi}) hrd\bar{\phi}$$

and the superbars indicate that the variable is a function of z and t. This reduction to Tso's equations is significant since elimination of the shear effects and shear equations in Tso's equation leads to Vlasov's equations [23], and elimination of the rotary inertia reduces Vlasov's equations to the well-known Gere equations for the coupled vibrations of thin-walled open cross-section beams [50].

Reiteration of the Simplifying Assumptions

The assumptions associated with the derivation of the equations of motion (3-60) to (3-66) are repeated for convenient reference. They are:

1. The ring is composed of a homogeneous, isotropic material.
2. The shell is thin ($h/r \leq 1/20$).
3. The deflections of the shell and their spatial derivatives are small and are considered linear.
4. The transverse normal stress is negligible.
5. Normals to the reference surface of the shell remain normal to it and undergo no change in length during deformation.
6. In-plane stretching dominates the bending effect ($\kappa_2 h \ll e_2$).
7. A given cross-section of the curved beam is undeformable; i.e. moves as a rigid body in its own plane.
8. Any contribution to the strain energy due to u_{2_s}, i.e. axial deflection due to shear, is neglected.
9. Rotary inertia of the shell about its midline has been neglected.
10. The ratio of the radius of the cross-section to the radius of the ring is small ($r/R \leq 1/5$).

11. All terms in the binomial series expansion terms in the strain energy expression due to bending that are multiplied by $(r/R)^n$ where $n \geq 3$ are considered relatively small and are truncated.

Orthogonality Relationship for the Shell to Ring Out-of-Plane Equations of Motion

The out-of-plane equations of motion for the complete or incomplete ring (3-63), (3-64), (3-65), and (3-66) in matrix form are

$$[M]\begin{Bmatrix} \ddot{w} \\ \ddot{r\psi} \end{Bmatrix} + [L]\begin{Bmatrix} w \\ r\psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3-79}$$

where

$$[M] = \begin{bmatrix} M_{21}-M_7D_2 & M_{21} & M_{23}+M_{22}D_2 & M_{23} \\ M_{21} & M_{21} & M_{23} & M_{23} \\ M_{23}+M_{22}D_2 & M_{23} & M_{24}-M_{25}D_2 & M_{24} \\ M_{23} & M_{23} & M_{24} & M_{24} \end{bmatrix}$$

$$[L] = \begin{bmatrix} E^*I_3D_4-\frac{1}{3}\,GI_{53}D_2 & -\frac{1}{3}\,GI_{47}D_2 & E^*I_{55}D_4-E^*I_{11}D_2-\frac{1}{3}\,GI_{54}D_2 & -E^*I_{11}D_2-\frac{1}{3}\,GI_{56}D_2 \\ -\frac{1}{3}\,GI_{47} & -\,GI_{57}D_2 & -\frac{1}{3}\,GI_{51}D_2 & -\,GI_{58}D_2 \\ E^*I_{55}D_4-E^*I_{11}D_2-\frac{1}{3}\,GI_{54}D_2 & -\frac{1}{3}\,GI_{51}D_2 & E^*I_{59}D_4-2E^*I_{60}D_2+E^*I_8-\frac{1}{3}\,GI_{61}D_2 & +E^*I_{60}D_2-\frac{G}{3}\,I_{63}D_2+E^*I_8 \\ -E^*I_{11}D_2-\frac{1}{3}\,GI_{56}D_2 & -GI_{51}D_2 & -E^*I_{60}D_2-\frac{1}{3}GI_{63}D_2+E^*I_8 & E^*I_8-GI_{62}D_2 \end{bmatrix}$$

and

$$(\eta) = \begin{Bmatrix} \eta_b \\ \eta_s \end{Bmatrix}, \quad (r\psi) = \begin{Bmatrix} r\psi_b \\ r\psi_s \end{Bmatrix}, \quad (0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The harmonic solution is

$$\begin{Bmatrix} \eta(\theta,t) \\ r\psi(\theta,t) \end{Bmatrix} = e^{i\omega_n t} \begin{Bmatrix} H_n(\theta) \\ \Psi_n(\theta) \end{Bmatrix} \qquad (3\text{-}80)$$

Substituting (3-80) into the equation of motion (3-79) gives

$$-\omega_n^2 [M] \begin{Bmatrix} H_n(\theta) \\ \Psi_n(\theta) \end{Bmatrix} + [L] \begin{Bmatrix} H_n(\theta) \\ \Psi_n(\theta) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3\text{-}81)$$

The orthogonality relationship is derived through the following procedure. Premultiplying (3-81) by $\lfloor H_n(\theta) \quad \Psi_n(\theta) \rfloor$ and integrating over the complete or incomplete ring gives

$$\int_0^{\theta_o} \lfloor H_n \quad \Psi_n \rfloor [L] \begin{Bmatrix} H_m \\ \Psi_m \end{Bmatrix} R d\theta = \omega_n^2 \int_0^{\theta_o} \lfloor H_n \quad \Psi_n \rfloor [M] \begin{Bmatrix} H_m \\ \Psi_m \end{Bmatrix} R d\theta \qquad (3\text{-}82)$$

Interchanging indices and subtracting one equation from the other yields

$$\int_0^{\theta_o} \lfloor H_m \quad \Psi_m \rfloor [L] \begin{Bmatrix} H_n \\ \Psi_n \end{Bmatrix} R d\theta - \int_0^{\theta_o} \lfloor H_n \quad \Psi_n \rfloor [L] \begin{Bmatrix} H_m \\ \Psi_m \end{Bmatrix} R d\theta =$$

$$\omega_m^2 \int_0^{\theta_o} \lfloor H_m \quad \Psi_m \rfloor [M] \begin{Bmatrix} H_n \\ \Psi_n \end{Bmatrix} R d\theta - \omega_n^2 \int_0^{\theta_o} \lfloor H_n \quad \Psi_n \rfloor [M] \begin{Bmatrix} H_m \\ \Psi_m \end{Bmatrix} R d\theta \qquad (3\text{-}83)$$

The right-hand side, R.H.S., of (3-83) after integration by parts may be written as

$$\begin{aligned} \text{R.H.S.} = {} & \omega_m^2 \left(M_{22} \frac{1}{R} \frac{\partial}{\partial\theta} \Psi_{b_n} - M_7 \frac{1}{R} \frac{\partial}{\partial\theta} H_{b_n} \right) H_{b_m} \Big|_0^{\theta_o} - \\ & \omega_n^2 \left(M_{22} \frac{1}{R} \frac{\partial}{\partial\theta} \Psi_{b_m} - M_7 \frac{1}{R} \frac{\partial}{\partial\theta} H_{b_m} \right) H_{b_n} \Big|_0^{\theta_o} + \\ & \omega_m^2 \left(M_{22} \frac{1}{R} \frac{\partial}{\partial\theta} H_{b_n} - M_{25} \frac{1}{R} \frac{\partial}{\partial\theta} \Psi_{b_n} \right) \Psi_{b_m} \Big|_0^{\theta_o} - \qquad (3\text{-}84) \\ & \omega_n^2 \left(M_{22} \frac{1}{R} \frac{\partial}{\partial\theta} H_{b_m} - M_{25} \frac{1}{R} \frac{\partial}{\partial\theta} \Psi_{b_m} \right) \Psi_{b_n} \Big|_0^{\theta_o} + \\ & (\omega_m^2 - \omega_n^2) \int_0^{\theta_o} \lfloor H_m \quad \Psi_m \rfloor [M^*] \begin{Bmatrix} H_n \\ \Psi_n \end{Bmatrix} R d\theta \end{aligned}$$

where

$$[M^*] = \begin{bmatrix} M_{21} & M_{21} & M_{23} & M_{23} \\ M_{21} & M_{21} & M_{23} & M_{23} \\ M_{23} & M_{23} & M_{24} & M_{24} \\ M_{23} & M_{23} & M_{24} & M_{24} \end{bmatrix}$$

Integrating the appropriate terms in the left-hand side, L.H.S., of (3-83) gives

$$\text{L.H.S.} = (\text{L.H.S.})_{mn} - (\text{L.H.S.})_{nm}$$

where

$$(\text{L.H.S.})_{mn} = E^*I_3[H_{b_m} D_3 H_{b_n} \Big|_0^{\theta_o} - D_1 H_{b_m} D_2 H_{b_n} \Big|_0^{\theta_o}] - \frac{1}{3} GI_{53}[H_{b_m} D_1 H_{b_n} \Big|_0^{\theta_o}] -$$

$$\frac{1}{3} GI_{47}[H_{b_m} D_1 H_{b_n} \Big|_0^{\theta_o}] + E^*I_{35}[H_{b_m} D_3 \Psi_{b_n} \Big|_0^{\theta_o} - D_1 H_{b_m} D_2 \Psi_{b_n} \Big|_0^{\theta_o}] -$$

$$E^*I_{11}[H_{b_m} D_1 \Psi_{b_n} \Big|_0^{\theta_o}] - \frac{1}{3} GI_{54}[H_{b_m} D_1 \Psi_{b_n} \Big|_0^{\theta_o}] -$$

$$[E^*I_{11} + \frac{1}{3} GI_{56}][H_{b_m} D_1 \Psi_{s_n} \Big|_0^{\theta_o}] - \frac{1}{3} GI_{47}[H_{s_m} D_1 H_{b_n} \Big|_0^{\theta_o}] -$$

$$GI_{57}[H_{s_m} D_1 H_{s_n} \Big|_0^{\theta_o}] - \frac{1}{3} GI_{51}[H_{s_m} D_1 \Psi_{b_n} \Big|_0^{\theta_o}] - GI_{58}[H_{s_m} D_1 \Psi_{s_n} \Big|_0^{\theta_o}] +$$

$$E^*I_{55}[\Psi_{b_m} D_3 H_{b_n}\Big|_0^{\theta_o} - D_1\Psi_{b_m} D_2 H_{b_n}\Big|_0^{\theta_o}] - E^*I_{11}[\Psi_{b_m} D_1 H_{b_n}\Big|_0^{\theta_o}] -$$

$$\frac{1}{3}\, GI_{54}[\Psi_{b_m} D_1 H_{b_n}\Big|_0^{\theta_o}] - \frac{1}{3}\, GI_{51}[\Psi_{b_m} D_1 H_{s_n}\Big|_0^{\theta_o}] +$$

$$E^*I_{59}[\Psi_{b_m} D_3 \Psi_{b_n}\Big|_0^{\theta_o} - D_1\Psi_{b_m} D_2 \Psi_{b_n}\Big|_0^{\theta_o}] - 2E^*I_{60}[\Psi_{b_m} D_1 \Psi_{b_n}\Big|_0^{\theta_o}] -$$

$$\frac{1}{3}\, GI_{61}[\Psi_{b_m} D_1 \Psi_{b_n}\Big|_0^{\theta_o}] - [E^*I_{60} + \frac{G}{3}\, I_{63}][\Psi_{b_m} D_1 \Psi_{s_n}\Big|_0^{\theta_o}] -$$

$$[E^*I_{11} + \frac{1}{3}\, GI_{56}][\Psi_{s_m} D_1 H_{b_n}\Big|_0^{\theta_o}] - GI_{58}[\Psi_{s_m} D_1 \Psi_{s_n}\Big|_0^{\theta_o}] -$$

$$[E^*I_{60} + \frac{G}{3}\, I_{63}][\Psi_{s_m} D_1 \bar{\Psi}_{b_n}\Big|_0^{\theta_o}] - GI_{62}[\Psi_{s_m} D_1 \Psi_{s_n}\Big|_0^{\theta_o}] \tag{3-85}$$

and

$$(L.H.S.)_{mn} = (L.H.S.)^T_{nm}$$

The L.H.S. plus the non-integral terms of the R.H.S. are zero for selected boundary conditions. The homogeneous boundary conditions usually encountered in practice are pinned ends, fixed ends, and the complete ring which has no ends. By pinned end it is meant that there is no displacement or bending moment, no rotation and no restraint of warping. By fixed end, it is meant that there is no displacement, no slope, no rotation, and no warping at the end. The complete ring requires continuity in the displacements and rotations and their first spatial derivatives at some arbitrary point, say the origin. Mathematically the boundary conditions are

Pinned End: $\eta_b = 0;\ \eta_b'' = 0;\ \eta_s = 0;\ \psi_b = 0,\ \psi_b'' = 0;\ \psi_s = 0$

Clamped End: $\eta_b = 0,\ \eta_b' = 0;\ \eta_s = 0;\ \psi_b = 0,\ \psi_b' = 0;\ \psi_s = 0$

Complete Ring:

$$\eta_b(\theta,t) = \eta_b(\theta+2\pi,t)$$

$$\eta_b'(\theta,t) = \eta_b'(\theta+2\pi,t)$$

$$\eta_s(\theta,t) = \eta_s(\theta+2\pi,t)$$

$$\psi_b(\theta,t) = \psi_b(\theta+2\pi,t)$$

$$\psi_b'(\theta,t) = \psi_b'(\theta+2\pi,t)$$

$$\psi_s(\theta,t) = \psi_s(\theta+2\pi,t)$$

The free end boundary conditions involve partial differential equations and are not considered here.

Subject to the above boundary conditions, (3-83) becomes

$$(\omega_m^2-\omega_n^2) \int_0^{\theta_o} \lfloor H_m \quad \Psi_m \rfloor [M^*] \begin{Bmatrix} H_n \\ \Psi_n \end{Bmatrix} R d\theta = 0 \qquad m \neq n$$

For non-degenerate modes, $\omega_m \neq \omega_n$ for $m \neq n$ and (3-85) can be normalized to give

$$\frac{\int_0^{\theta_o} \lfloor H_m \quad \Psi \rfloor [M^*] \begin{Bmatrix} H_n \\ \Psi_n \end{Bmatrix} R d\theta}{\int_0^{\theta_o} \lfloor H_m \quad \Psi_m \rfloor \begin{Bmatrix} H_m \\ \Psi_m \end{Bmatrix} R d\theta} = \delta_{mn} \tag{3-86}$$

where δ_{mn} is the Kronecker delta.

CHAPTER IV

COMPARISON OF CONVENTIONAL THEORY TO SHELL-TO-RING THEORY

Introduction

The solutions of the shell-to-ring equations for free oscillations are presented in this chapter for the complete ring. This solution is compared with the eigenfrequencies of the conventional solution using a variety of thicknesses and radii ratios. As included in the derivation in Chapter III, the ring is restricted so that its plane of symmetry lies in the plane of the ring. This restriction splits the problem into in-plane and out-of-plane vibrations, and it is felt that this lack of coupling will not distract from the overall comparisons.

Eigenfrequencies of In-Plane Complete Ring Vibrations for the Shell-to-Ring Theory

The equations of motion for the in-plane free oscillations of the complete ring are

$$- M_5 \ddot{\zeta} + E^*I_1 \frac{1}{R^2} \zeta'' + M_9 \frac{1}{R} \ddot{\xi}_b' + E^*I_5 \frac{1}{R^3} \xi_b''' + E^*I_9 \frac{1}{R} \xi_b'$$

$$+ E^*I_9 \frac{1}{R} \xi_s' = 0 \qquad (3\text{-}60)$$

$$M_9 \frac{1}{R} \ddot{\zeta}' + E^*I_5 \frac{1}{R^3} \zeta''' + E^*I_9 \frac{1}{R} \zeta' + M_{13} \ddot{\xi}_b - M_6 \frac{1}{R^2} \ddot{\xi}_b'' + E^*I_2 \frac{1}{R^4} \xi_b''''$$

$$+ 2E^*I_{10} \frac{1}{R^2} \xi_b'' + E^*I_7\xi_b + M_{13}\ddot{\xi}_s + E^*I_{10} \frac{1}{R^2} \xi_s''$$

$$+ E^*I_7\xi_s = 0. \tag{3-61}$$

$$E^*I_9 \frac{1}{R} \zeta' + M_{13}\ddot{\xi}_b + E^*I_{10} \frac{1}{R^2} \xi_b'' + E^*I_7\xi_b + M_{13}\ddot{\xi}_s +$$

$$E^*I_7\xi_s - GI_{52} \frac{1}{R^2} \xi_s'' = 0 \tag{3-62}$$

with boundary conditions based on continuity of the deflections and their slopes so that no tears or wrinkles developed in the vibrating ring. Such boundary conditions are

$$\zeta(\theta,t) = \zeta(\theta+2\pi,t)$$

$$\xi_b(\theta,t) = \xi_b(\theta+2\pi,t)$$

$$\xi_b'(\theta,t) = \xi_b'(\theta+2\pi,t)$$

$$\xi_s(\theta,t) = \xi_s(\theta+2\pi,t)$$

The anti-symmetric radial terms and the symmetric axial terms of the trigonometric Fourier series satisfy the equations of motion and are

$$\zeta_n(\theta,t) = Z_n \sin n\theta e^{i\omega_n t} \qquad n = 1,2,3,\ldots$$

$$\xi_{b_n}(\theta,t) = \Xi_{b_n} \cos n\theta e^{i\omega_n t} \qquad n = 0,1,2,\ldots \tag{4-1}$$

$$\xi_{s_n}(\theta,t) = \Xi_{s_n} \cos n\theta e^{i\omega_n t} \qquad n = 0,1,2,\ldots$$

Substituting the solution (4-1) into the equation of free oscillations (3-60), (3-61), and (3-62) and rearranging the resulting homogeneous algebraic equations gives the characteristic equation for $n > 0$ from which the eigenfrequencies may be determined. That equation in determinant form is given on the following page.

The frequency corresponding to the breathing mode is obtained for the special case when $n = 0$. The solution (4-1) is substituted into (3-60), (3-61), and (3-62) and then n is set equal to zero. The resulting equations are

$$(-M_{20}\omega_o^2 \Xi_b + E^*I_7 \Xi_b - M_{20}\omega_o^2 \Xi_s + E^*I_7 \Xi_s)e^{i\omega_o t} = 0$$

$$(-M_{20}\omega_o^2 \Xi_b + E^*I_7 \Xi_b - M_{20}\omega_o^2 \Xi_s + E^*I_7 \Xi_s)e^{i\omega_o t} = 0$$

These equations may be written as

$$(E^*I_7 - M_{20}\omega_o^2)(\Xi_b + \Xi_s)e^{i\omega_o t} = 0 \tag{4-3}$$

The only non-trivial solution of this equation is the relationship

$$\begin{vmatrix} E^*I_1\left(\frac{n}{R}\right)^2 - M_5\omega_n^2 & E^*I_5\left(\frac{n}{R}\right)^3 - E^*I_9\left(\frac{n}{R}\right) + M_9\left(\frac{n}{R}\right)\omega_n^2 & -E^*I_9\left(\frac{n}{R}\right) \\ E^*I_5\left(\frac{n}{R}\right)^3 - E^*I_9\left(\frac{n}{R}\right) + M_9\left(\frac{n}{R}\right)\omega_n^2 & E^*I_2\left(\frac{n}{R}\right)^4 - 2E^*I_{10}\left(\frac{n}{R}\right)^2 + E^*I_7 - M_{13}\omega_n^2 - M_6\left(\frac{n}{R}\right)^2\omega_n^2 & -E^*I_{10}\left(\frac{n}{R}\right)^2 + E^*I_7 - M_{13}\omega_n^2 \\ -E^*I_9\left(\frac{n}{R}\right) & -E^*I_{10}\left(\frac{n}{R}\right)^2 + E^*I_7 - M_{13}\omega_n^2 & E^*I_7 + GI_{52}\left(\frac{n}{R}\right)^2 - M_{13}\omega_n^2 \end{vmatrix} = 0 \qquad (4\text{-}2)$$

$$\omega_o^2 = \frac{E^* I_7}{M_{20}}$$

$$= \frac{E^*}{\rho R^2} \frac{\int_{-\alpha}^{\alpha} [1 - a\cos\phi]^{-1} d\phi}{\int_{-\alpha}^{\alpha} [1 - a\cos\phi] d\phi} \qquad (4\text{-}4)$$

<u>Eigenfrequencies of Out-of-Plane Complete</u>
<u>Ring Vibrations for the Shell-to-Ring Theory</u>

The eigenvalue problem for the out-of-plane free oscillations of the shell-to-ring theory is

$$M_{14}\ddot{\eta}_b - M_7 \frac{1}{R^2} \ddot{\eta}_b'' + E^* I_3 \frac{1}{R^4} \eta_b'''' - \frac{1}{3} GI_{53} \frac{1}{R^2} \eta_b'' + M_{14}\ddot{\eta}_s - \frac{1}{3} GI_{47} \frac{1}{R^2} \eta_s'' +$$

$$M_{16} r\ddot{\psi}_b + M_{15} \frac{r}{R^2} \ddot{\psi}_b'' - \frac{1}{3} GI_{54} \frac{r}{R^2} \psi_b'' + E^* I_{55} \frac{r}{R^4} \psi_b'''' - E^* I_{11} \frac{r}{R^2} \psi_b'' +$$

$$M_{16} r\ddot{\psi}_s - E^* I_{11} \frac{r}{R^2} \psi_s'' - \frac{1}{3} GI_{56} \frac{r}{R^2} \psi_s'' = 0 \qquad (3\text{-}63)$$

$$M_{14}\ddot{\eta}_b - \frac{1}{3} GI_{47} \frac{1}{R^2} \eta_b'' + M_{14}\ddot{\eta}_s - GI_{57} \frac{1}{R^2} \eta_s'' + M_{16} r\ddot{\psi}_b - \frac{1}{3} GI_{51} \frac{r}{R^2} \psi_b'' +$$

$$M_{16} r\ddot{\psi}_s - GI_{58} \frac{r}{R^2} \psi_s'' = 0 \qquad (3\text{-}64)$$

$$M_{16}\ddot{\eta}_b + M_{15} \frac{1}{R^2} \ddot{\eta}_b'' - \frac{1}{3} GI_{54} \frac{1}{R^2} \eta_b'' + E^* I_{55} \frac{1}{R^4} \eta'''' - E^* I_{11} \frac{1}{R^2} \eta_b'' +$$

$$M_{16}\ddot{\eta}_s - \frac{1}{3} GI_{51} \frac{1}{R^2} \eta_s'' + M_{17} r\ddot{\psi}_b - M_{18} \frac{r}{R^2} \ddot{\psi}_b'' + E^* I_{59} \frac{r}{R^4} \psi_b'''' -$$

$$2E^*I_{60}\,\frac{r}{R^2}\,\psi_b'' + E^*I_8 r\psi_b - \frac{1}{3}\,GI_{61}\,\frac{r}{R^2}\,\psi_b'' + M_{17}r\ddot{\psi}_s - E^*I_{60}\,\frac{r}{R^2}\,\psi_s'' +$$

$$E^*I_8 r\psi_s - \frac{1}{3}\,GI_{63}\,\frac{r}{R^2}\,\psi_s'' = 0 \tag{3-65}$$

$$M_{16}\ddot{\eta}_b - E^*I_{11}\,\frac{1}{R^2}\,\eta_b'' - \frac{1}{3}\,GI_{56}\,\frac{1}{R^2}\,\eta_b'' + M_{16}\ddot{\eta}_s - GI_{58}\,\frac{1}{R^2}\,\eta_s'' + M_{17}r\ddot{\psi}_b -$$

$$\frac{G}{3}\,I_{63}\,\frac{r}{R^2}\,\psi_b'' - E^*I_{60}\,\frac{r}{R^2}\,\psi_b'' + E^*I_8 r\psi_b + M_{17}r\ddot{\psi}_s + E^*I_8 r\psi_s -$$

$$GI_{62}\,\frac{r}{R^2}\,\psi_s'' = 0 \tag{3-66}$$

with complete ring boundary conditions

$$\eta_b(\theta,t) = \eta_b(\theta+2\pi,t)$$

$$\eta_b'(\theta,t) = \eta_b'(\theta+2\pi,t)$$

$$\eta_s(\theta,t) = \eta_s(\theta+2\pi,t)$$

$$\psi_b(\theta,t) = \psi_b(\theta+2\pi,t)$$

$$\psi_b'(\theta,t) = \psi_b'(\theta+2\pi,t)$$

$$\psi_s(\theta,t) = \psi_s(\theta+2\pi,t)$$

The symmetric terms or the antisymmetric terms of the trigonometric series satisfy the equations of motion (3-63), (3-64), (3-65), and (3-66). The anti-symmetric terms of the solution are

$$
\begin{aligned}
\eta_{b_n}(\theta,t) &= H_{b_n} \sin n\theta e^{i\omega_n t}, && n = 1,2,3,\ldots \\
\eta_{s_n}(\theta,t) &= H_{s_n} \sin n\theta e^{i\omega_n t} && n = 1,2,3,\ldots \\
r\psi_{b_n}(\theta,t) &= \Psi_{b_n} \sin n\theta e^{i\omega_n t} && n = 1,2,3,\ldots \\
r\psi_{s_n}(\theta,t) &= \Psi_{s_n} \sin n\theta e^{i\omega_n t} && n = 1,2,3,\ldots
\end{aligned}
\tag{4-5}
$$

Substituting this solution into the equations of motion and rearranging results in the characteristic equation in determinant form on the following page.

Comparison of Elementary and Higher Order Theories

The characteristic equations of the pertinent solutions in Chapter II and the preceding sections of this chapter were programmed in Fortran IV for use on the Univac 1108 digital computer. Comparisons are outlined and presented in the following sections.

In-Plane Modes

The in-plane "conventional" theory and the shell-to-ring theory where the shear effects and shear equations and the rotary inertia effects are neglected were compared to the shell-to-ring theory.

$$
\begin{vmatrix}
E^*I_3\left(\frac{n}{R}\right)^4 + \frac{1}{3}GI_{53}\left(\frac{n}{R}\right)^2 - M_{14}\omega_n^2 - M_7\left(\frac{n}{R}\right)^2\omega_n^2 & \frac{1}{3}GI_{47}\left(\frac{n}{R}\right)^2 - M_{14}\omega_n^2 & \frac{1}{3}GI_{54}\left(\frac{n}{R}\right)^2 + E^*I_{51}\left(\frac{n}{R}\right)^4 + E^*I_{11}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 + M_{15}\left(\frac{n}{R}\right)^2\omega_n^2 & E^*I_{11}\left(\frac{n}{R}\right)^2 + \frac{1}{3}GI_{56}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 \\
\frac{1}{3}GI_{47}\left(\frac{n}{R}\right)^2 - M_{14}\omega_n^2 & GI_{57}\left(\frac{n}{R}\right)^2 - M_{14}\omega_n^2 & \frac{1}{3}GI_{51}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 & GI_{58}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 \\
\frac{1}{3}GI_{54}\left(\frac{n}{R}\right)^2 + E^*I_{51}\left(\frac{n}{R}\right)^4 + E^*I_{11}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 + M_{15}\left(\frac{n}{R}\right)^2\omega_n^2 & \frac{1}{3}GI_{51}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 & E^*I_{59}\left(\frac{n}{R}\right)^4 + 2E^*I_{60}\left(\frac{n}{R}\right)^2 + E^*I_8 + \frac{1}{3}GI_{61}\left(\frac{n}{R}\right)^2 - M_{17}\omega_n^2 - M_{18}\left(\frac{n}{R}\right)^2\omega_n^2 & E^*I_{60}\left(\frac{n}{R}\right)^2 + E^*I_8 + \frac{1}{3}GI_{63}\left(\frac{n}{R}\right)^2 - M_{17}\omega_n^2 \\
E^*I_{11}\left(\frac{n}{R}\right)^2 + \frac{1}{3}GI_{56}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 & GI_{58}\left(\frac{n}{R}\right)^2 - M_{16}\omega_n^2 & E^*I_{60}\left(\frac{n}{R}\right)^2 + E^*I_8 + \frac{1}{3}GI_{63}\left(\frac{n}{R}\right)^2 - M_{17}\omega_n^2 & E^*I_8 + GI_{62}\left(\frac{n}{R}\right)^2 - M_{17}\omega_n^2
\end{vmatrix} = 0 \qquad (4\text{-}6)
$$

These comparisons are plots of frequency versus number of nodes and are given in Figures 5, 6, 9, and 10. The values of the pertinent geometric parameters are labeled in the figures. Solutions were obtained using the integrals evaluated with Simpson's rule and the binomial series expansion but the results were found to be in such good agreement for the radii ratio considered that they are presented as one curve.

It is noted from Figures 5 and 9 that when the radii ratio is small, that is

$$\frac{r}{R} = a << 1$$

then the influences of shear and rotary inertia are indeed insignificant for in-plane eigenfrequencies. It must be pointed out, however, that the shear center is in the plane of the ring and so far as in-plane vibrations are concerned the ring is symmetric, and the potential influences of an unsymmetric ring on in-plane eigenmodes is not discussed in this thesis.

Figures 6 and 10 reflect the influences of the shear and rotary inertia for a significant increase in the radii ratio, but for a ratio that remains within the limitations of the binomial series expansion solution. That is,

$$\frac{r}{R} = a = \frac{1}{R} .$$

The only difference in these two figures is the angle of cut-out, $2\phi_o$, and since the curves are seen to agree qualitatively, the effect of ϕ_o on in-plane vibrations is found to be negligible. Additional studies involving thickness variations were conducted and it was found that thickness influences stiffness and mass equally and therefore exhibits no control over the frequency. The thickness must be restricted, however, to comply with thin-shell theory limitations.

Both Figures 6 and 10 reflect the fact that a ring with a radii ratio as large as one-fifth exhibits geometric influences that force the conventional solution and the shell-to-ring solutions without shear and rotary inertia to diverge as the number of nodes in the free oscillations increases. The added effects of shear and rotary inertia increase that divergence in an anticipated fashion. Finally it is pointed out that the influence of the inextensional assumption reflected in the Hoppe solution is smaller than might be expected for the magnitudes of the geometric parameters involved.

Bending Predominant Modes

Out-of-plane comparisons consist of "conventional" theory, Krahula's results, Michell's results, and the shell-to-ring theory without shear equations and rotary inertia effects versus the shell-to-ring theory. These shell-to-ring solutions include both the Simpson's rule solution and the binomial series expansion solution but are represented as one curve since the results of these two methods are again in very good agreement for the ranges of the radii ratio considered. These comparisons are given in Figure 7, 8, 11, and 12.

The conventional solution and the various shell-to-ring solutions are seen to converge for a decreasing radii ratio as exhibited in Figures 7 and 11. All of these solutions account for the eccentricity of the shear center from the center of gravity. Krahula's solution neglects this effect and is seen to be in considerable error for as few as eight nodes and a radii ratio as low as one to five.

Figures 8 and 12 show the increasing effects of the shear and rotary inertia as the radii ratio approaches the limitations of the binomial series expansion solution. The neglect of the shear center eccentricity by Krahula is seen to be appreciable here even for the lowest possible free oscillation of the complete ring.

Michell's solution while coinciding with the results of Krahula is in fact different. Michell neglected warping and used the polar moment of inertia for his torsional constant. Consequently his solution is not applicable and is included merely for comparison and completeness and is not recommended for contemporary studies of thin-walled open-section rings.

The shell-to-ring theory has the restriction of inextensionality placed on the cross-sectional in-plane deflection, u_1. That is,

$$\sigma_{22} = \frac{E}{1-\nu^2}[\varepsilon_2+\nu\varepsilon_1]$$

for thin shell theory (see p. 44, [42]), but when the restriction that the shell cross-section translates and rotates as a rigid body is made, then

$$\varepsilon_1 = 0. \tag{3-27a}$$

Hence,

$$\sigma_{22} = \frac{E}{1-\nu^2}\,\varepsilon_2$$

instead of the beam relationship

$$\sigma_{22} = E\varepsilon_2.$$

This effect tends to increase the stiffness and can result in higher eigenfrequencies for the shell-to-ring theory without shear effects and rotary inertia than for the conventional theory. The inclusion of shear effects and rotary inertia lowers the frequencies dramatically as the number of nodes increases.

Breathing Mode

The breathing frequency for the shell-to-ring theory is non-dimensionalized with respect to the Timoshenko frequency, ω_T, [38] and is plotted versus the radii ratio in Figure 13. Here again the inextensional restriction on the in-plane cross-section forces the shell-to-ring eigenfrequency to be higher than the results of Timoshenko. This is the $\nu = 1/3$ curve in Figure 13. The breathing frequency without the inextensional restriction can be obtained by solving (4-4) for ω_o/ω_T where $\nu = 0$. A plot of this frequency appears in Figure 13, and thus a comparison of solutions with and without the inextensional

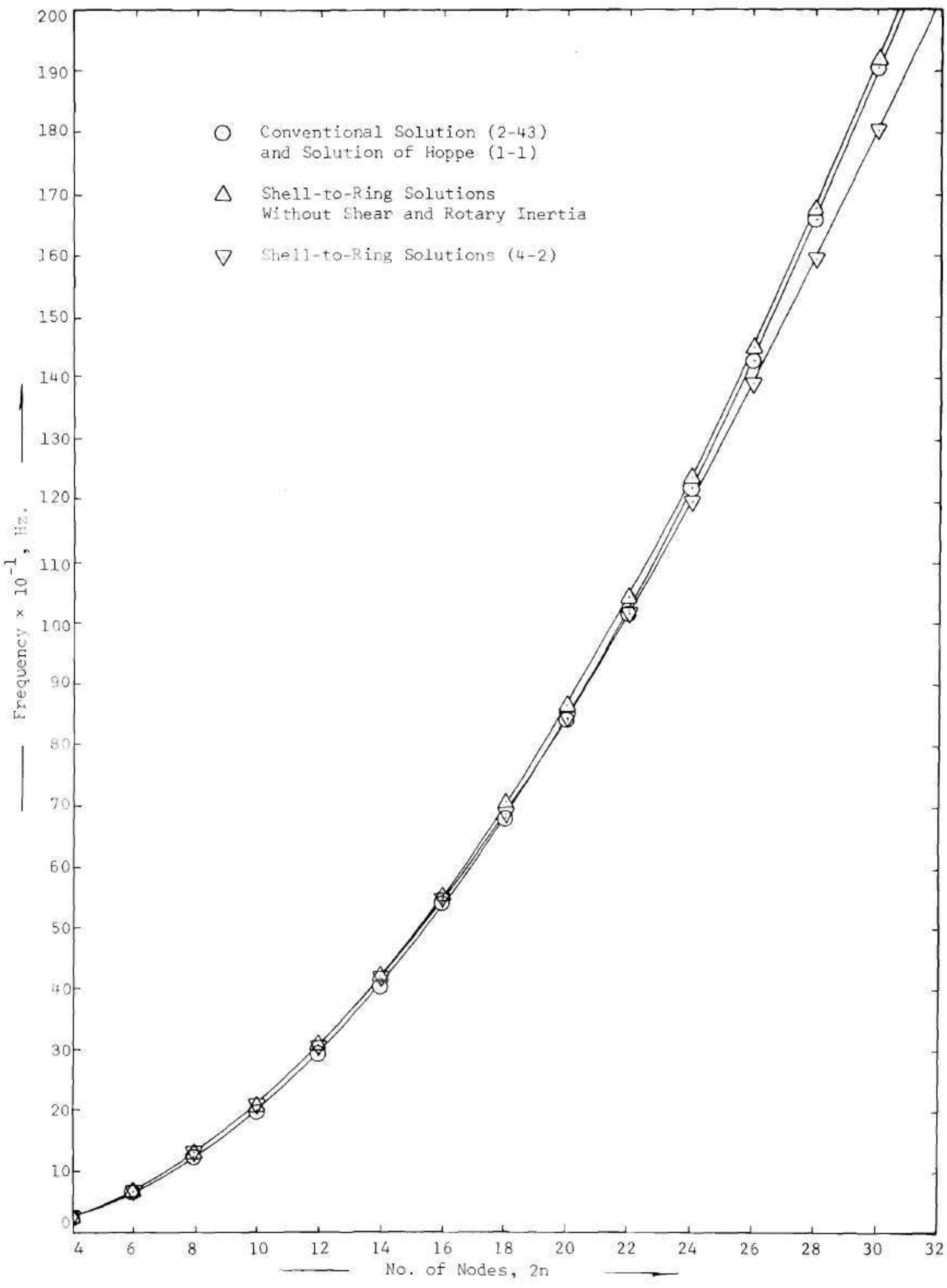

Figure 5. Comparisons of the Various Solutions for the In-Plane Eigenfrequencies of an Aluminum Ring with r = 1", R = 50", h = 0.05", and ϕ_o = 10°

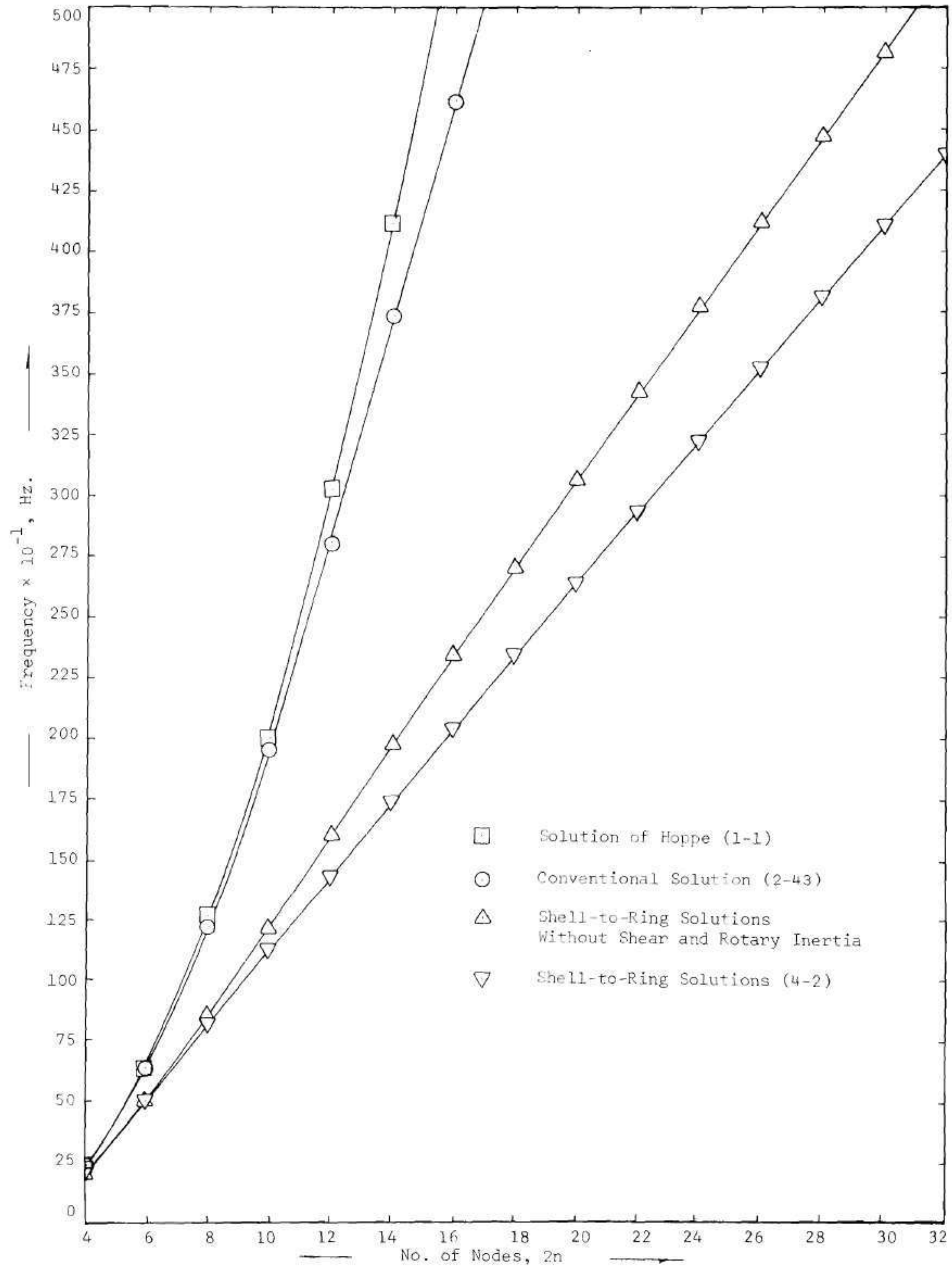

Figure 6. Comparisons of the Various Solutions for the In-Plane Eigenfrequencies of an Aluminum Ring with r = 10", R = 50", h = 0.50", and ϕ_o = 10°

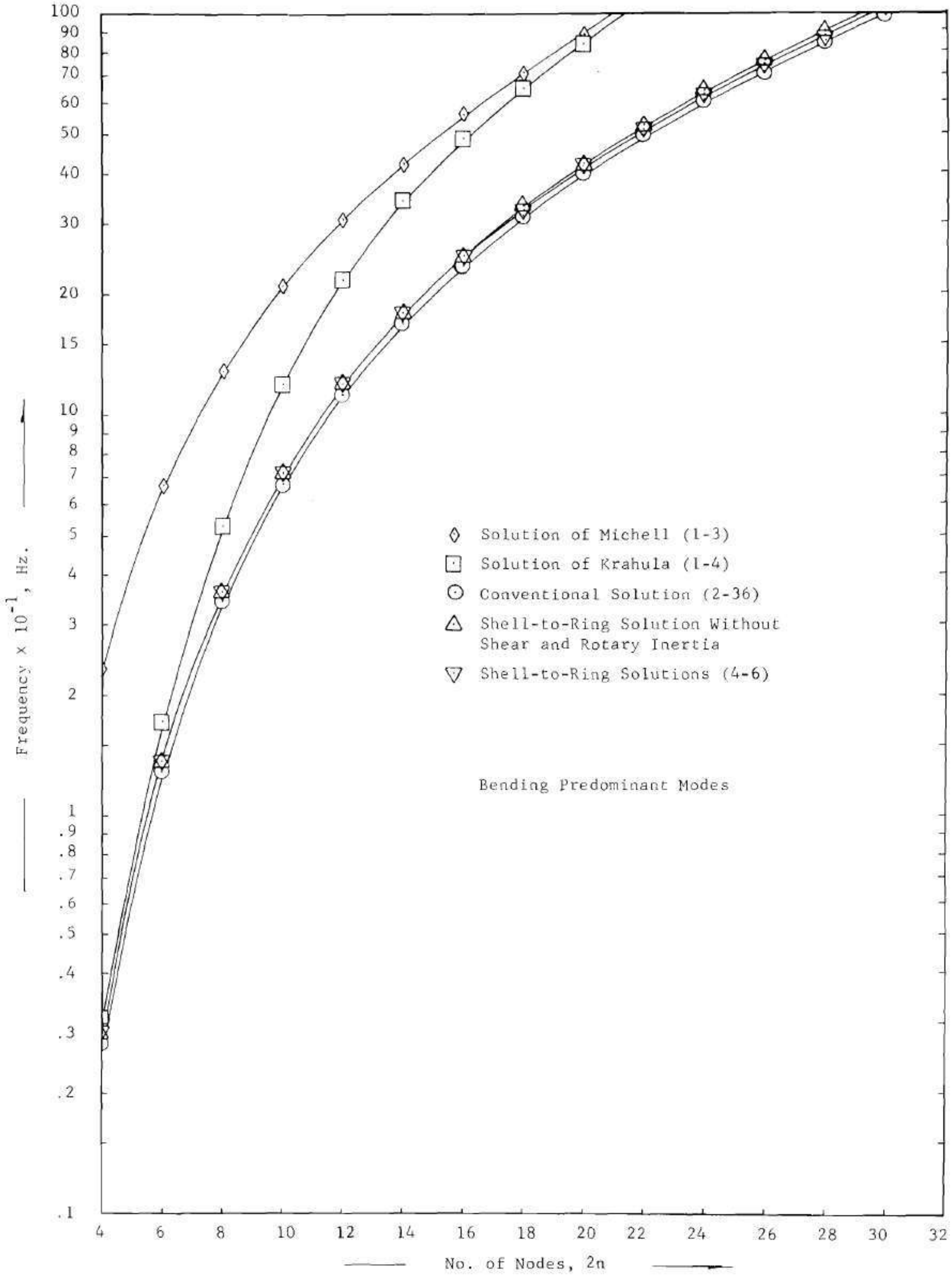

Figure 7. Comparisons of the Various Solutions for the Out-of-Plane Eigenfrequencies of an Aluminum Ring with r = 1", R = 50", h = 0.05", and ϕ_o = 10°

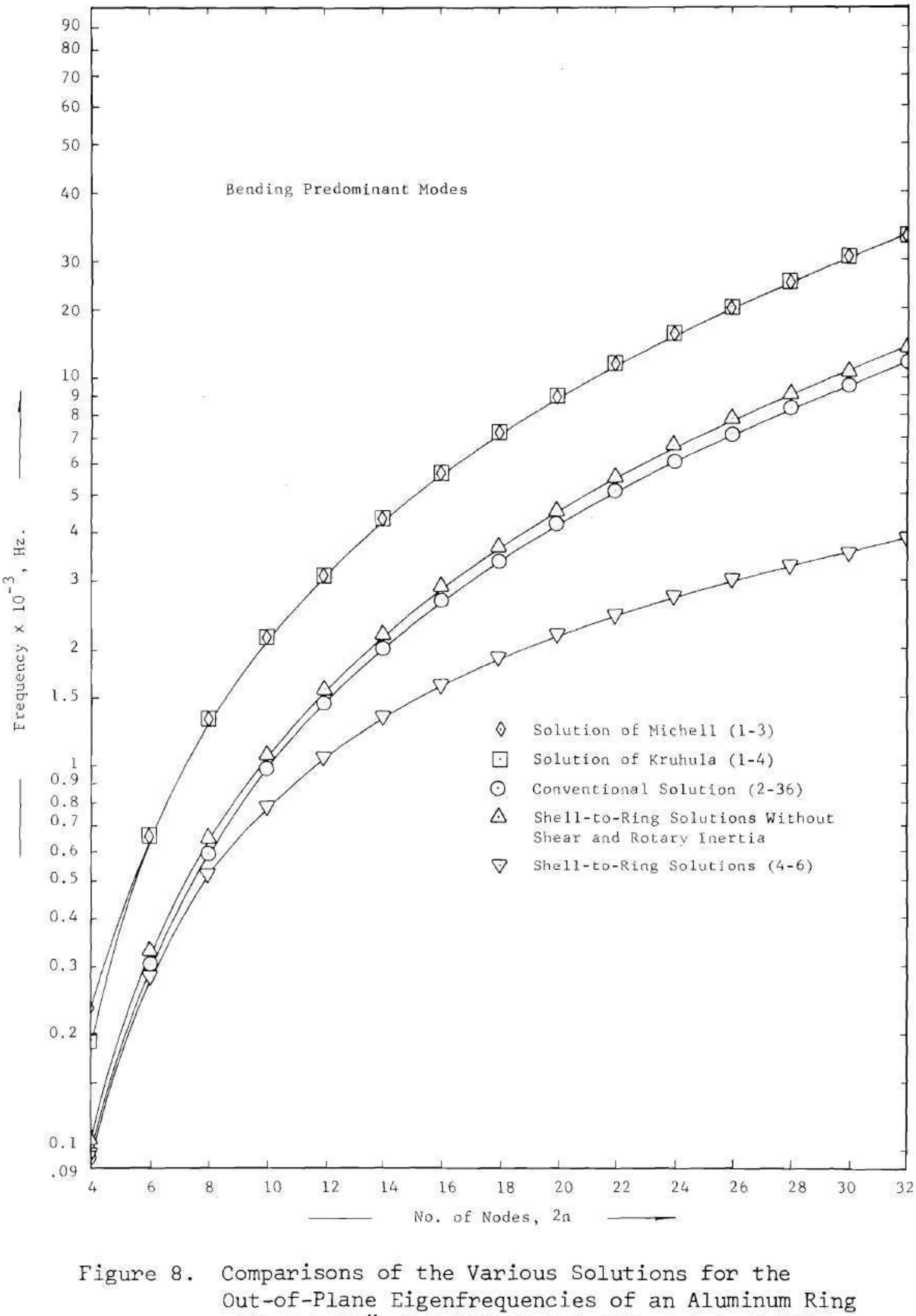

Figure 8. Comparisons of the Various Solutions for the Out-of-Plane Eigenfrequencies of an Aluminum Ring with r = 10", R = 50", h = 0.50", and ϕ_o = 10°

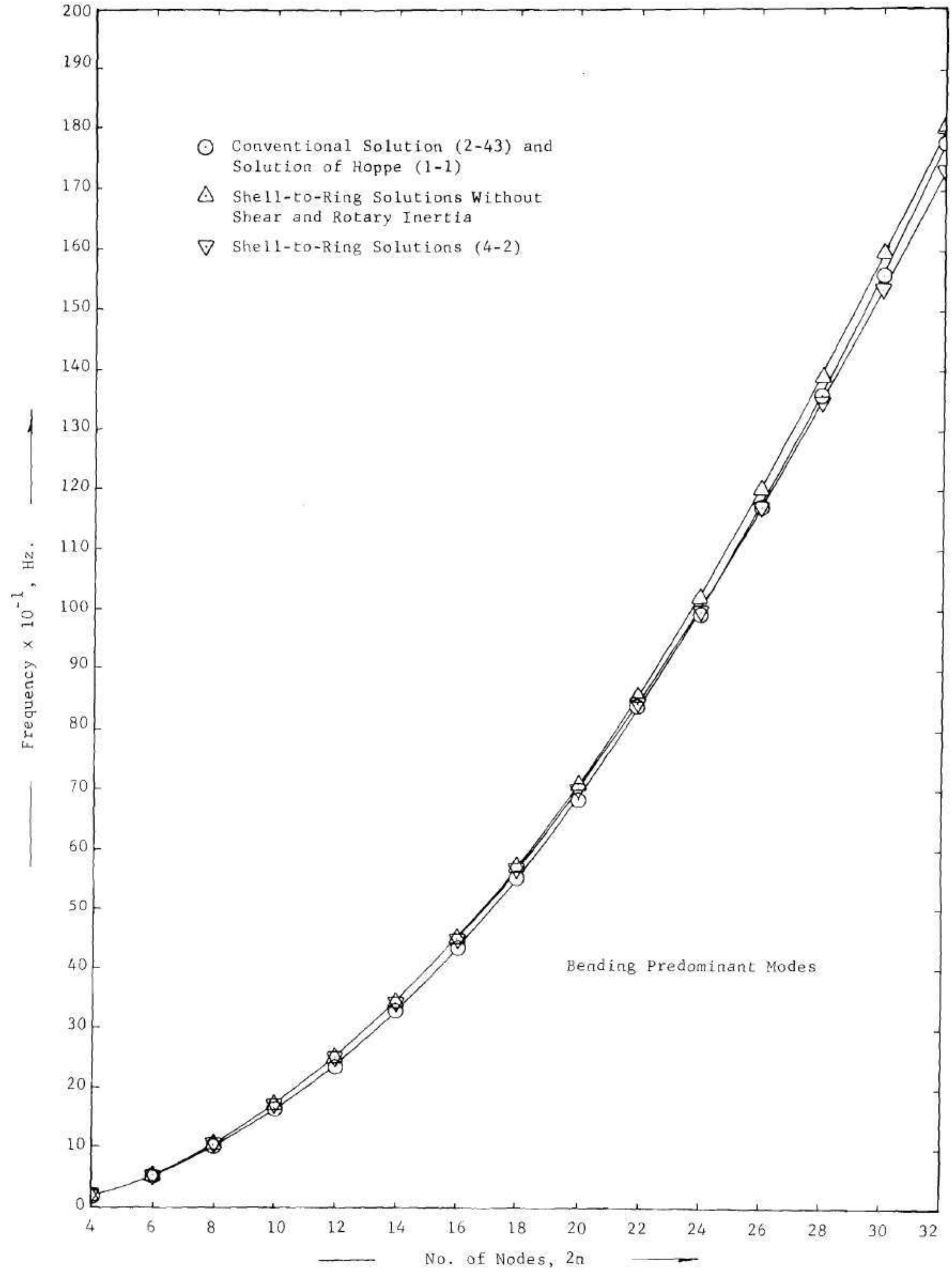

Figure 9. Comparisons of the Various Solutions for In-Plane Eigenfrequencies of an Aluminum Ring with r = 1", R = 50", h = 0.05", and ϕ_o = 45°

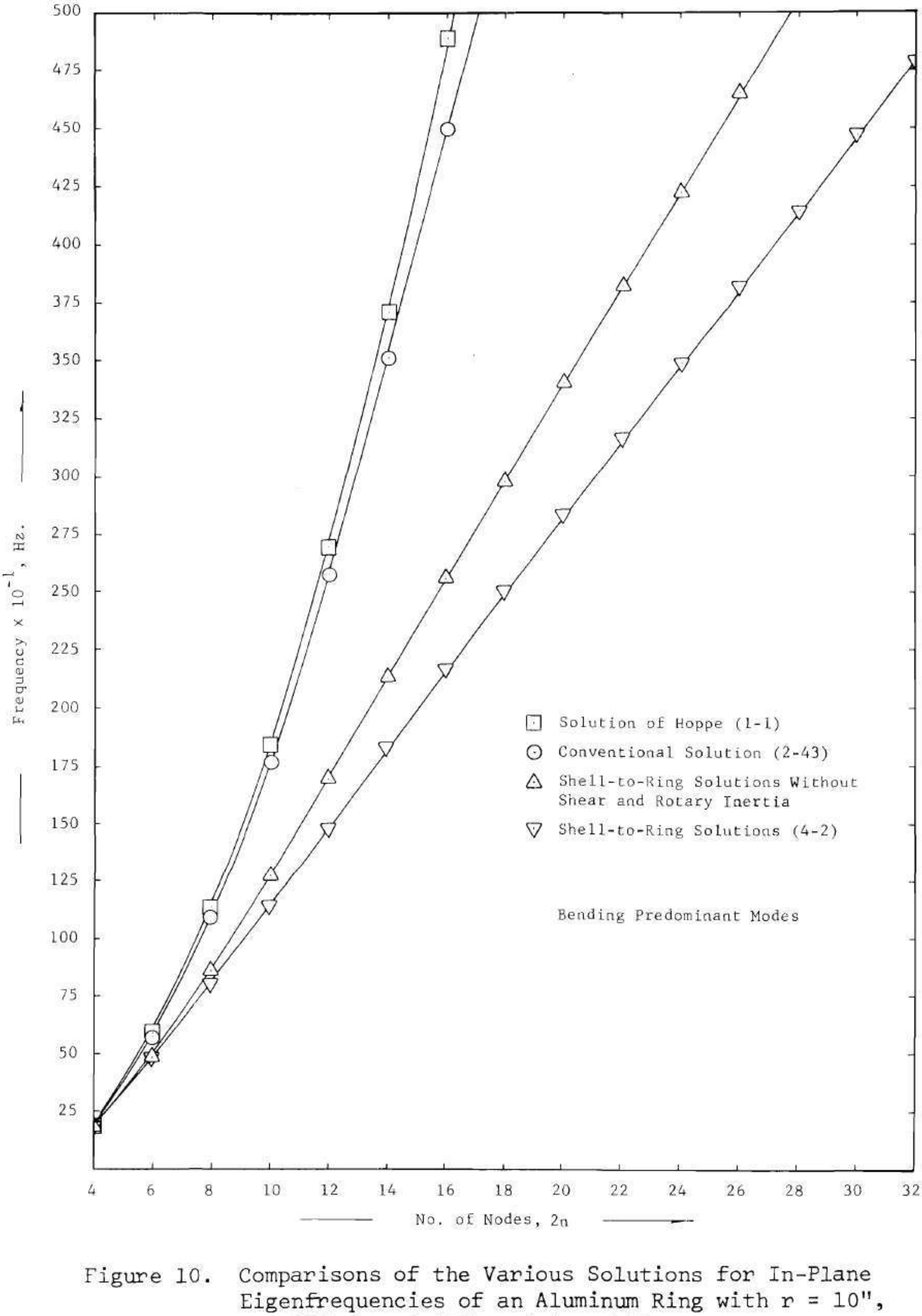

Figure 10. Comparisons of the Various Solutions for In-Plane Eigenfrequencies of an Aluminum Ring with r = 10", R = 50", h = 0.50", and ϕ_o = 45°

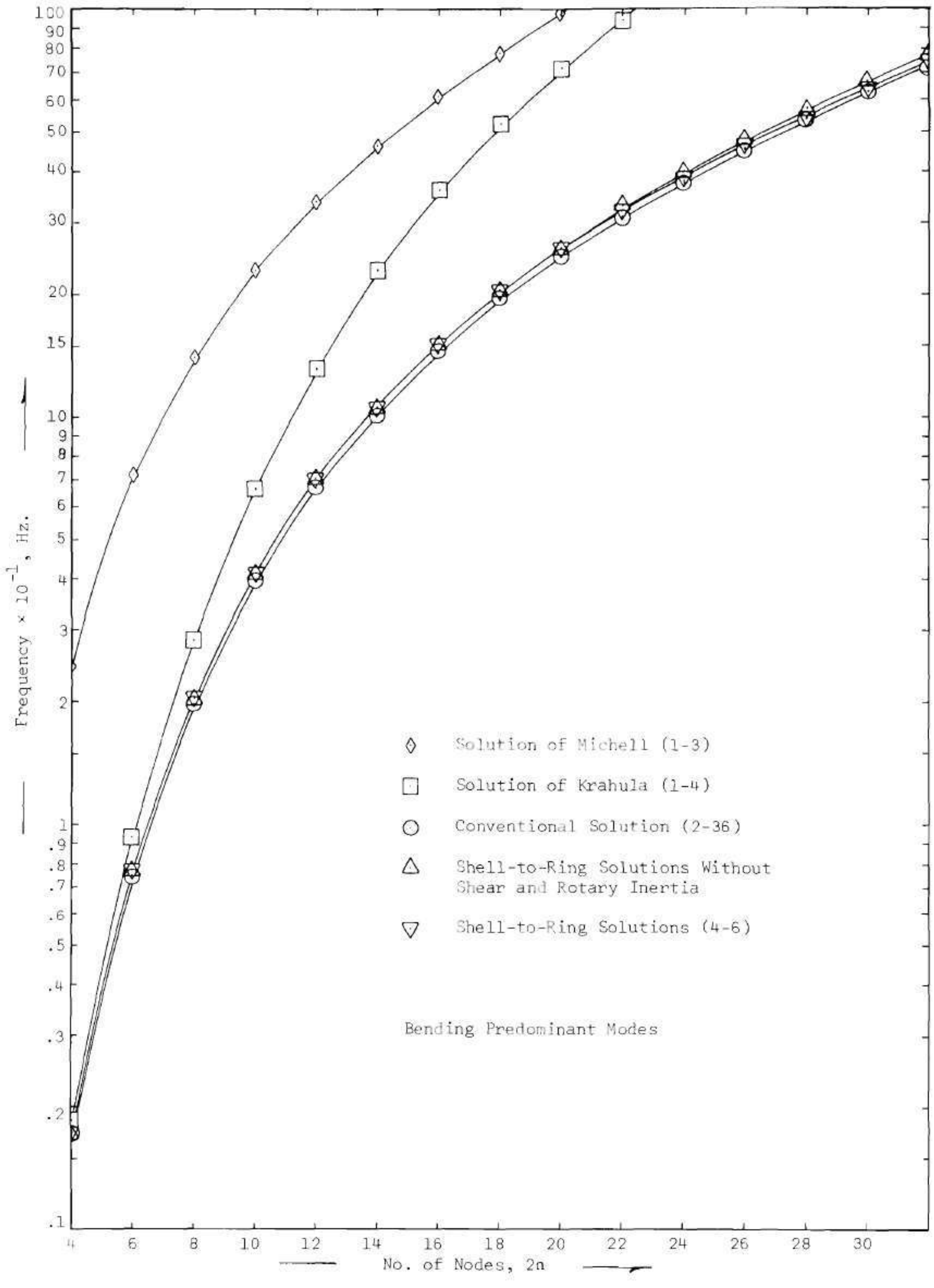

Figure 11. Comparisons of the Various Solution for Out-of-Plane Eigenfrequencies of an Aluminum Ring with r = 1", R = 50", h = 0.05", and ϕ_o = 45°

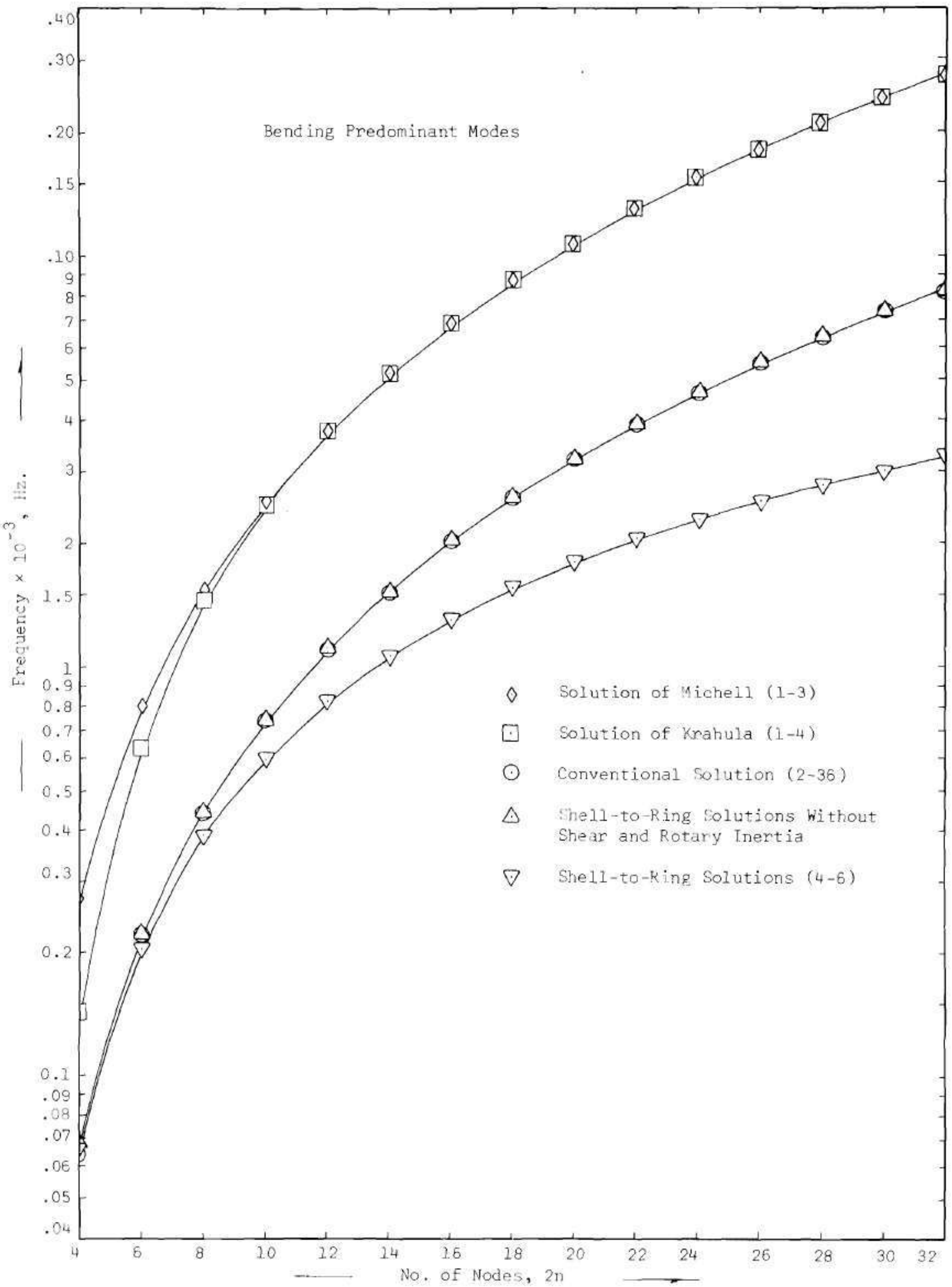

Figure 12. Comparisons of the Various Solutions for Out-of-Plane Eigenfrequencies of an Aluminum Ring with r = 10", R = 50", h = 0.50", and ϕ_o = 45°

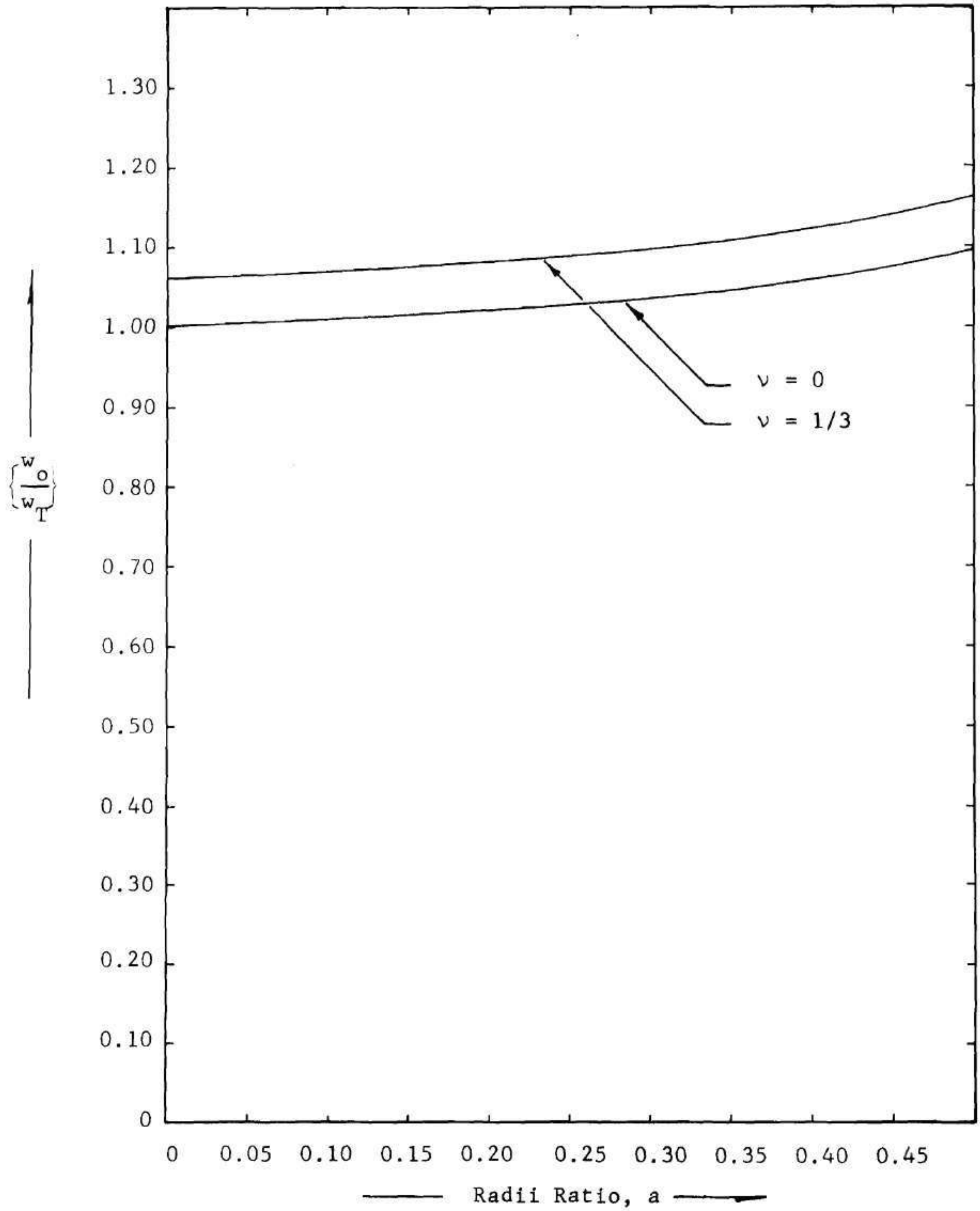

Figure 13. Shell-To-Ring Breathing Frequency Nondimensionalized With Respect to Timoshenko Breathing Frequency Versus Radii Ratio.

assumption is readily available for the given geometry. It should also be noted that the geometric influences of the ring neglected in Timoshenko's analysis tend to increase frequency by 10 per cent for a radii ratio of one-half.

Torsion Predominant Modes

In most cases and in the case of the slit-tubular rings considered here, the lowest possible eigenfrequency corresponding to the rotation dominated eigenmode is usually higher than the first few eigenfrequencies corresponding to the translation dominated eigenmodes. That is, the lowest rotational eigenfrequency is usually out of the range of the translational eigenfrequencies that can be reasonably represented by a theory that neglects shear effects and rotary inertia. Consequently in an elementary analysis the rotation of the cross-section included through the polar moment of inertia can be neglected without loss of accuracy of the translational eigenfrequencies.

The range of applicability of the translation dominated eigenmodes corresponding to the shell-to-ring theory, however, includes the first few rotation dominated eigenmodes. These eigenfrequencies are therefore worth some comment.

Comparisons of the eigenfrequencies corresponding to the rotation dominated eigenmodes are given in Figures 14 and 15. These frequencies are the second lowest eigenfrequencies obtained in the shell-to-ring frequency studies and are the higher frequencies obtained in the conventional vibration solution and in Krahula's solution.

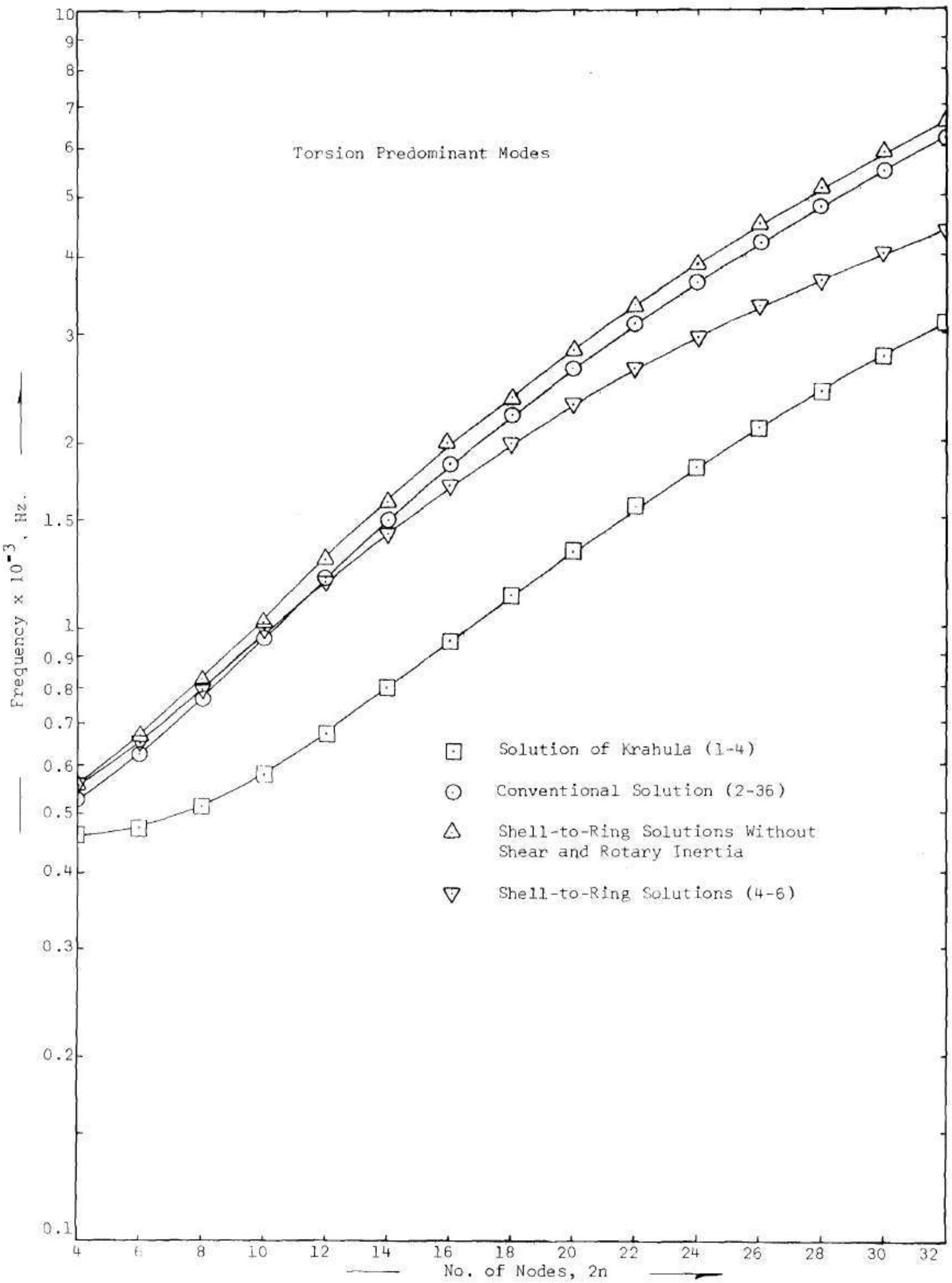

Figure 14. Comparisons of the Various Solutions for Eigenfrequencies Corresponding to Rotationally Predominant Eigenmodes for an Aluminum Ring with r = 1", R = 50", h = 0.05", and ϕ_o = 10°

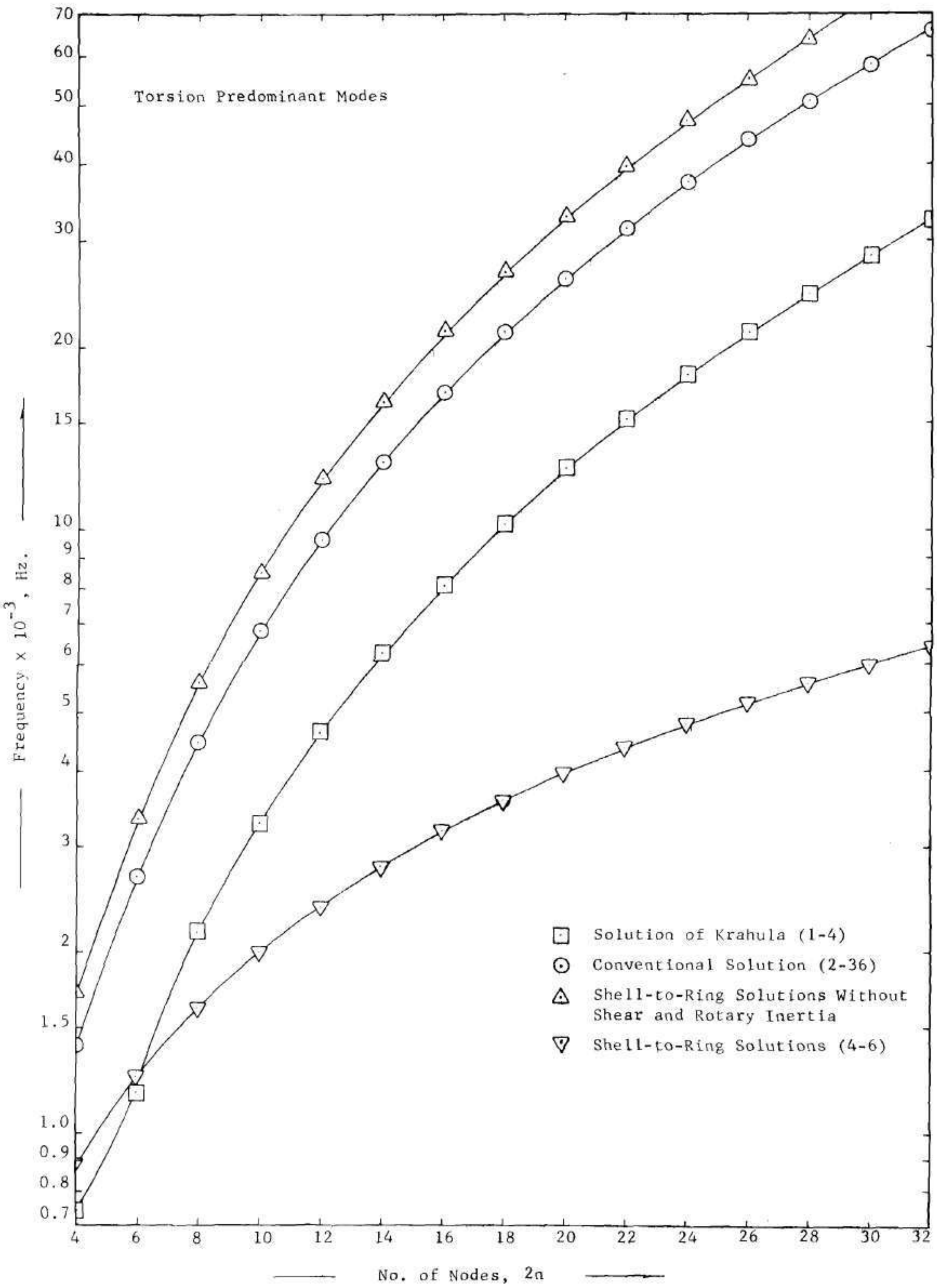

Figure 15. Comparisons of the Various Solutions for Eigenfrequencies Corresponding to Rotationally Predominant Eigenmodes for an Aluminum Ring with $r = 10''$, $R = 50''$, $h = 0.50''$, and $\phi_o = 10°$

The primary difference between Figures 14 and 15 is in the radii ratio. Figure 14 represents a ring having a radii ratio of 1 to 50 and can be considered as having a comparatively small cross-sectional area or a relatively long circumferential length. Here the shear effects and rotary inertia terms have less influence than in the ring of Figure 15 where the radii ratio is one to five. It should be noted, however, that the influences of shear and rotary inertia on the eigenfrequencies of the rotation dominated eigenmodes are much greater than on the eigenfrequencies of the translation dominated eigenmodes. This is attributed to the higher energy levels of these high frequency low number-of-node eigenmodes and is analogous to the straight beam results found by Tso [37].

The frequencies found by Krahula are for the most part much lower than the corresponding frequencies of the other theories for the rings represented in Figures 14 and 15.

CHAPTER V

CONCLUSIONS

The four conventional equations of motion for the free oscillations of a ring with a thin-walled open monosymmetric cross-section have been developed including the effects of warping, St. Venant torsion, extensionality, and a shear center eccentricity from the center of gravity. With the plane of symmetry lying in the plane of the ring these equations split into two coupled sets of equations: one for in-plane oscillations, the other for out-of-plane oscillations.

The out-of-plane ring equations have been found to reduce to ring equations of motion available in the literature by elimination of various parameters. Assuming that the shear center eccentricity is zero gives the equations of motion developed by Krahula [29]; further assumptions that the torsional inertia and the warping effects are negligible reduces the equations to those of Michell [19] with appropriate modifications for the St. Venant torsional coefficient.

Studies of these various systems of equations of motion have shown that the effect of the shear center eccentricity on the natural frequencies of rings with radii ratios as low as 1 to 50 is not negligible. Hence, a vibration study involving monosymmetric rings should include the eccentricity of the shear center if a "reasonable" prediction of natural frequencies and/or the response of a system to a given forcing function is being sought. As would be expected, the further neglect of the warping gives completely erroneous results for thin-walled rings.

The coupled in-plane equations of motion for a symmetric ring are documented in the literature for the inextensional case and for the case of inextensional curvature terms--extensional circumferential terms. The pure extensional solution, which leads to a characteristic equation that is the determinant of a non-symmetric matrix, has been found to give eigenfrequencies that agree to three decimal places with the eigenfrequencies of the equations of motion based on the inextensional curvature relations for ranges of parameters under consideration in this study. Hence it seems plausible to conclude that the eigenfrequencies of the equations of motion based on inextensional curvature relations are satisfactorily close to the eigenfrequencies of the pure extensional solution for thin-walled rings. The advantage is that the equations of motion based on inextensional curvature relations are symmetric.

The eigenfrequencies of the shell-to-ring equations of motion for the free oscillations of one particular monosymmetric ring whose plane of symmetry lies in the plane of the ring have been compared with the eigenfrequencies of the conventional equations of motion analyzed above.

It is felt that generalizations can be made to extend the results to thin-walled rings of open cross-section such as channels or I sections.

The shell-to-ring equations are based on the assumption that the cross-sectional length is inextensional. The consequencies of this assumption are best seen in the breathing mode where the frequency is higher due to this assumption by the Poisson ratio effect $1/(1-\nu^2)$. This same effect is found in many of the terms of the in-plane and out-of-plane equations of motion, but an isolation of the effects of this particular assumption for the eigenfrequencies in general is not possible.

An interesting comparison is made between the eigenfrequencies of the conventional solution and the eigenfrequencies of the shell-to-ring solution where the shear effects and rotary inertia terms were neglected for in-plane and out-of-plane modes.

The in-plane differences are seen to be 2 to 3 per cent for radii ratios such as one to five. The out-of-plane differences are seen to be 2 or 3 per cent for all of the radii ratios considered.

Thickness variations in the ring were found to have little or no effect, but it should be kept in mind that the shell-to-ring theory is based on shell theory where the thickness to radius ratio is assumed to be in the neighborhood of 1 to 20.

Variations in the angle of cutout were found to produce no qualitative change in the frequency studies.

Examination of the torsionally predominant modes for the shell-to-ring theory shows an increased importance for the shear effects and rotary inertia terms for a given mode shape. For radii ratios as large as one to five these differences are extreme. It should be pointed out, however, that the eigenfrequency for a given eigenmode is much higher for a torsionally predominant mode than for a bending predominant mode. Consequently the torsion predominant modes are of less importance in a structural dynamics study of a system involving rings and thin members such as a fuselage, large missile, or ship hull.

Comparisons of the conventional eigenfrequencies to those for the shell-to-ring theory for bending predominant eigenmodes shows that for very small radii ratios, i.e. 1 to 50, the shear and rotary inertia are of little or no importance up to and including 32 node points. For radii ratios as large as one to five, however, the shear and rotary inertia effects are appreciable for ten or more nodes. Extrapolating these results to apply to the channel section, it seems safe to say that shear effects and rotary inertia terms have little bearing on vibration studies involving eigenfrequencies that correspond to less than 30 nodes for aircraft such as the C-5A or C-141 because of the very low radii ratios involved. Vehicles such as the Saturn booster, however, involve larger radii ratios and hence it is concluded that shear effects and rotary inertia could have a considerable bearing on eigenfrequencies and resulting vibration studies.

APPENDIX

APPENDIX A

CALCULATION OF THE TIMOSHENKO WARPING CONSTANT

The coefficient of warping as defined by Timoshenko [32] is calculated for the slit tubular cross-section for use in the conventional out-of-plane free vibration study of Chapter II.

The warping coefficient is defined as

$$C_w = \int_{-\beta}^{\beta} (D-W_s)^2 hr d\phi \tag{A-1}$$

where

$$D = \frac{1}{2\beta} \int_{-\beta}^{\beta} \int_{-\beta}^{\phi} h(\phi) r d\phi r d\phi$$

$$W_s = \int_{-\beta}^{\beta} h(\phi) r d\phi$$

$$h(\phi) = (x-a_x)\cos\phi - (y-a_y)\sin\phi \tag{3-7}$$

$$(x-a_x) = (R-a_x) - r\cos\phi = e - r\cos\phi$$

$$(y-a_y) = r\sin\phi - a_y \tag{3-11}$$

and where e is the distance from the center of the slit circular tube (not the center of gravity) to the shear center as defined in Figure A-1.

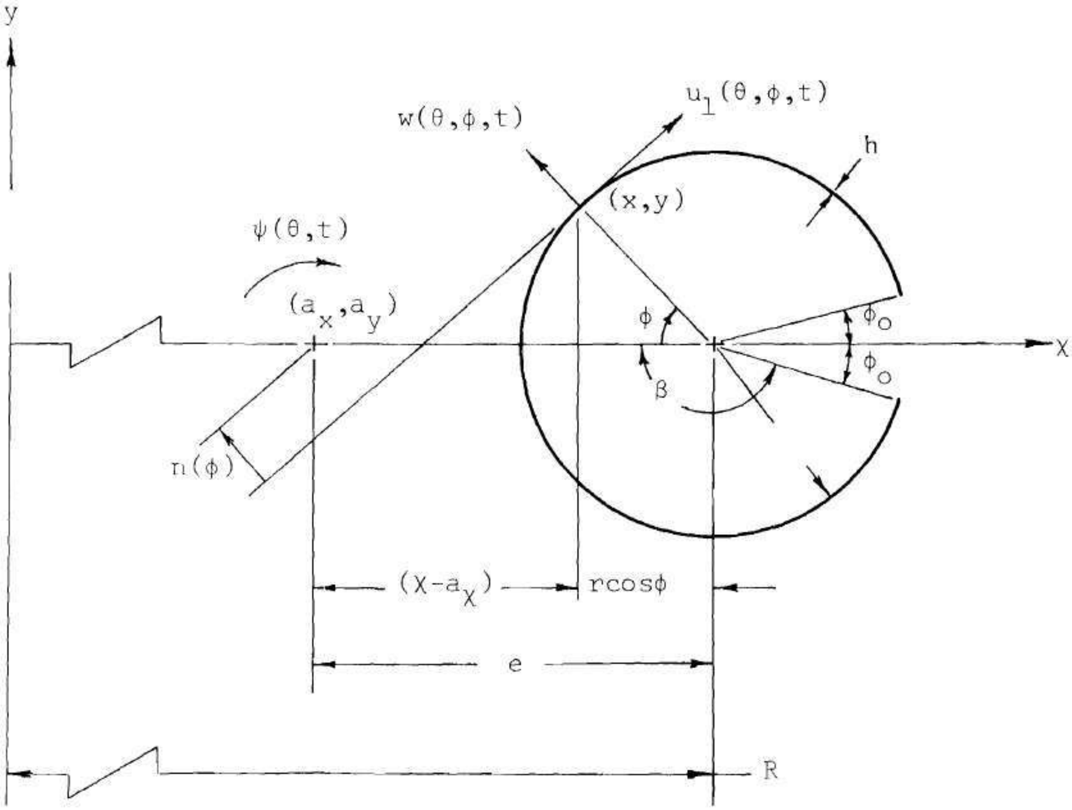

Figure A-1. Parameters Utilized in Calculation of Timoshenko Warping Constant

That distance can be expressed as

$$e = 2r\left\{\frac{\sin\beta - \beta\cos\beta}{\beta - \sin\beta\cos\beta}\right\} \tag{A-2}$$

Performing the integrations as described in (A-1) gives the warping constant for the slit tubular cross-section as

$$C_w = \frac{2}{3}hr^5\left[\beta^3 - \frac{6(\sin\beta - \beta\cos\beta)^2}{\beta - \sin\beta\cos\beta}\right] \tag{A-3}$$

APPENDIX B

STIFFNESS AND MASS INTEGRALS

The integrals indicated by the coefficients I_i, $i=1,2,\ldots,45$, and M_j, $j=1,2,\ldots,12$, in (3-58) are defined exactly for the Simpson's rule solution and approximately for the integrand binomial series expansion solution, respectively, below. Where only one definition appears the integral is defined as the same for both methods of solution. The I_i integrals result from stiffness terms and the M_j integrals result from the inertia terms.

$$I_1 = hr \frac{1}{(1-a)^2} \int_{-\alpha}^{\alpha} (1-a\cos\phi)d\phi$$

$$I_2 = hrR^2 \int_{-\alpha}^{\alpha} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right]^2 (1-a\cos\phi)d\phi$$

$$I_3 = hr^3 \int_{-\alpha}^{\alpha} \frac{4}{(1+a)^4} \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)} + \left(\frac{a}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\sqrt{\frac{1+a}{1-a}}\,\tan\left(\frac{\phi}{2}\right)\right]\right]^2 (1-a\cos\phi)d\phi.$$

$$\approx hr^3 \int_{-\alpha}^{\alpha} [\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\phi+2)]^2(1-a\cos\phi)d\phi$$

$$I_4 = hr^3 \int_{-\alpha}^{\alpha} \frac{4}{(1+a)^4} \frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)[(1-a) + (1+a)\tan^2(\frac{\phi}{2})]} +$$

$$\left[\left(\frac{1}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan\left(\frac{\phi}{2}\right)\right]\right]^2 (1-a\cos\phi)d\phi$$

$$\approx hr^3 \int_{-\alpha}^{\alpha} [\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)]^2(1-a\cos\phi)d\phi$$

$$I_5 = \frac{-hrR}{(1-a)} \int_{-\alpha}^{\alpha} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right](1-a\cos\phi)d\phi$$

$$I_6 = hr^3 \frac{4}{(1+a)^4} \int_{-\alpha}^{\alpha} \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} +\right.$$

$$\left(\frac{a}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan\left(\frac{\phi}{2}\right)\right] \left[\frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} +\right.$$

$$\left.\left(\frac{1}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan\left(\frac{\phi}{2}\right)\right]\right](1-a\cos\phi)d\phi$$

$$\approx hr^3 \int_{-\alpha}^{\alpha} [\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\phi+2)] \times$$

$$[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)](1-a\cos\phi)d\phi$$

$$I_7 = \frac{hr}{R^2} \int_{-\alpha}^{\alpha} \frac{d\phi}{(1-a\cos\phi)}$$

$$I_8 = \frac{hr}{R^2} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi d\phi}{(1-a\cos\phi)}$$

$$I_9 = \frac{hr}{R} \frac{1}{(1-a)} \int_{-\alpha}^{\alpha} d\phi$$

$$I_{10} = (-)hr \int_{-\alpha}^{\alpha} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right] d\phi$$

$$I_{11} = \frac{hr^2}{R} \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \sin\phi \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} + \right.$$

$$\left. \left(\frac{a}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right] d\phi$$

$$\simeq \frac{hr^2}{R} \int_{-\alpha}^{\alpha} \sin\phi[\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\phi+2)]d\phi$$

$$I_{12} = \frac{hr^2}{R} \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \sin\phi \left[\frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} + \right.$$

$$\left. \left(\frac{1}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right] d\phi$$

$$\simeq \frac{hr^2}{R} \int_{-\alpha}^{\alpha} \sin\phi[\phi + 2a\sin\phi + \frac{3}{2} a^2(\phi + \frac{1}{2}\sin 2\phi)]d\phi$$

$$I_{13} = hr \int_{-\alpha}^{\alpha} \frac{\sin^2\phi d\phi}{(1-a\cos\phi)}$$

$$I_{14} = hr \int_{-\alpha}^{\alpha} \frac{\cos^2\phi d\phi}{(1-a\cos\phi)}$$

$$I_{15} = hr \int_{-\alpha}^{\alpha} \frac{d\phi}{(1-a\cos\phi)}$$

$$I_{16} = hr \int_{-\alpha}^{\alpha} \frac{\cos\phi d\phi}{(1-a\cos\phi)}$$

$$I_{17} = \frac{h^3}{r} \int_{-\alpha}^{\alpha} \frac{d\phi}{(1-a\cos\phi)}$$

$$I_{18} = \frac{h^3 r}{R^4} \frac{1}{(1-a)^2} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi d\phi}{(1-a\cos\phi)}$$

$$I_{19} = \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right]^2 d\phi$$

$$I_{20} = \frac{h^3}{R} \left(\frac{r}{R}\right)^3 \frac{4}{(1+a)^4} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)]} + \right.$$

$$\left. \frac{a}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right]^2 d\phi$$

$$\simeq \frac{h^3}{R} \left(\frac{r}{R}\right)^3 \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} [\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) +$$

$$a^2\sin\phi(\cos^2\phi+2)]^2 d\phi$$

$$I_{21} = \frac{h^3 r}{R^3(1-a)} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right] d\phi$$

$$I_{22} = \frac{h^3}{R} \left(\frac{r}{R}\right)^3 \frac{4}{(1+a)^4} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} \left[\frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)]} + \right.$$

$$\left.\left(\frac{1}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right]^2 d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^3 \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)}[\phi + 2a\sin\phi + \frac{3}{2} a^2(\phi + \frac{1}{2} \sin 2\phi)]^2 d\phi$$

$$I_{23} = \frac{h^3 r^3}{R^4} \frac{4}{(1+a)^4} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} + \right.$$

$$\left.\frac{a}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+2}{1-2}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right] \left[\frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]}\right.$$

$$\left. + \frac{1}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right] d\phi$$

$$\simeq \frac{h^3 r^3}{R^4} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{(1-a\cos\phi)} [\sin\phi + a(\phi + \frac{1}{2} \sin 2\phi) +$$

$$a^2\sin\phi(\cos^2\phi+2)][\phi + 2a\sin\phi + \frac{3}{2} a^2(\phi + \frac{1}{2} \sin 2\phi)]d\phi$$

$$I_{24} = \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\cos^2\phi \sin^2\phi}{[1-a\cos\phi]^3}\, d\phi$$

$$I_{25} = \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\cos^4\phi\, d\phi}{[1-a\cos\phi]^3}$$

$$I_{26} = \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\cos^2\phi\, d\phi}{[1-a\cos\phi]^3}$$

$$I_{27} = \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\cos^3\phi\, d\phi}{[1-a\cos\phi]^3}$$

$$I_{28} = \frac{h^3 r}{R^2} \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos\phi}{[1-a\cos\phi]} \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} + \right.$$

$$\left. \frac{a}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right] d\phi$$

$$= \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos\phi}{(1-a\cos\phi)} [\sin\phi + a(\phi + \tfrac{1}{2}\sin 2\phi) + a^2 \sin\phi(\cos^2\phi + 2)]\, d\phi$$

$$I_{29} = \frac{h^3 r}{R^2} \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos\phi}{[1-a\cos\phi]} \left[\frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} + \right.$$

$$\left. \frac{1}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right] d\phi$$

$$\simeq \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos\phi}{[1-a\cos\phi]} [\phi + 2a\sin\phi + \frac{3}{2} a^2(\phi + \frac{1}{2} \sin 2\phi)]d\phi$$

$$I_{30} = \frac{h^3}{R} \int_{-\alpha}^{\alpha} \frac{\cos^2\phi d\phi}{[1-a\cos\phi]^2}$$

$$I_{31} = \frac{h^3}{R} \int_{-\alpha}^{\alpha} \frac{\cos\phi d\phi}{[1-a\cos\phi]^2}$$

$$I_{32} = \frac{h^3 r}{R^3(1-a)} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{[1-a\cos\phi]^2} d\phi$$

$$I_{33} = \frac{h^3 r}{R^2} \int_{-\alpha}^{\alpha} \frac{\sin^2\phi\cos^2\phi}{[1-a\cos\phi]^2} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right]d\phi$$

$$I_{34} = \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos^3\phi}{[1-a\cos\phi]^2} \left[\frac{(1+a)\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a) + (1+a)\tan^2\left(\frac{\phi}{2}\right)\right]} + \right.$$

$$\left. \frac{a}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan\left(\frac{\phi}{2}\right)\right]\right]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos^3\phi}{[1-a\cos\phi]^2} [\sin\phi + a(\phi + \frac{1}{2} \sin 2\phi) +$$

$$a^2 \sin\phi(\cos^2\phi+2)]d\phi$$

$$I_{35} = \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \frac{2}{(1+a)^2}\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos^3\phi}{[1-a\cos\phi]^2}\left[\frac{(1+a)a\tan\left(\frac{\phi}{2}\right)}{(1-a)\left[(1-a)+(1+a)\tan^2\left(\frac{\phi}{2}\right)\right]}+\right.$$

$$\left.\frac{1}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan\left(\frac{\phi}{2}\right)\right]\right]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^2\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos^3\phi}{[1-a\cos\phi]^2}\;[\phi + 2a\,\sin\phi + \frac{3}{2}\,a^2(\phi + \frac{1}{2}\,\sin2\phi)]d\phi$$

$$I_{36} = \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \frac{2}{(1+a)^2}\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos^2\phi}{[1-a\cos\phi]^2}\left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a)+(1-a)\tan^2(\phi/2)]}+\right.$$

$$\left.\frac{a}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\right]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^2\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos^2\phi}{[1-a\cos\phi]^2}\,[\sin\phi + a(\phi + \frac{1}{2}\,\sin2\phi) + a^2\sin\phi(\cos^2\phi+2)]d\phi$$

$$I_{37} = \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \frac{2}{(1+a)^2}\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos^2\phi}{[1-a\cos\phi]^2}\left[\frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a)+(1+a)\tan^2(\phi/2)]}+\right.$$

$$\left.\frac{1}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan\left(\frac{\phi}{2}\right)\right]\right]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos^2\phi}{[1-a\cos\phi]^2}\left[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)\right]d\phi$$

$$I_{38} = \frac{h^3}{R}\left(\frac{r}{R}\right) \int_{-\alpha}^{\alpha} \frac{\sin^4\phi d\phi}{[1-a\cos\phi]^3}$$

$$I_{39} = \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{\sin^3\phi\cos\phi}{[1-a\cos\phi]^2}\left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} + \left(\frac{a}{1+a}\right)\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\right]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \int_{-\alpha}^{\alpha} \frac{\sin^3\phi\cos\phi}{[1-a\cos\phi]^2}\left[\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\phi+2)\right]d\phi$$

$$I_{40} = \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{\sin^3\phi\cos\phi}{[1-a\cos\phi]^2}\left[\frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} + \frac{1}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\right]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)^2 \int_{-\alpha}^{\alpha} \frac{\sin^3\phi\cos\phi}{[1-a\cos\phi]^2}\left[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)\right]d\phi$$

$$I_{41} = \frac{h^3}{R}\left(\frac{r}{R}\right) \frac{2}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{\sin\phi\cos\phi}{[1-a\cos\phi]}\left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} + \right.$$

$$\frac{a}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\Bigg]d\phi$$

$$\simeq \frac{h^3}{R}\left(\frac{r}{R}\right)\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos\phi}{[1-a\cos\phi]}[\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(\cos^2\phi+2)]d\phi$$

$$I_{42} = \frac{h^3}{R}\left(\frac{r}{R}\right)\frac{2}{(1+a)^2}\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos\phi}{[1-a\cos\phi]}\Bigg[\frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a)+(1+a)\tan^2(\phi/2)]} +$$

$$\frac{1}{(1+a)}\left(\frac{1+a}{1-a}\right)^{3/2}\tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2}\tan(\phi/2)\right]\Bigg]d\phi$$

$$\simeq \frac{h^3}{R}\frac{r}{R}\int_{-\alpha}^{\alpha}\frac{\sin\phi\cos\phi}{[1-a\cos\phi]}[\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)]d\phi$$

$$I_{43} = \frac{h^3}{R}\int_{-\alpha}^{\alpha}\frac{\sin^2\phi}{[1-a\cos\phi]^2}\,d\phi$$

$$I_{44} = \frac{h^3}{R}\left(\frac{r}{R}\right)\int_{-\alpha}^{\alpha}\frac{\sin^2\phi\cos\phi}{[1-a\cos\phi]^3}\,d\phi$$

$$M_1 = \rho hr\int_{-\alpha}^{\alpha}\sin^2\phi[1-a\cos\phi]d\phi$$

$$M_2 = \rho hr\int_{-\alpha}^{\alpha}\cos^2\phi[1-a\cos\phi]d\phi$$

$$M_3 = \rho hr\int_{-\alpha}^{\alpha}[1-a\cos\phi]d\phi$$

$$M_4 = \rho hr \int_{-\alpha}^{\alpha} \cos\phi[1-a\cos\phi]d\phi$$

$$M_5 = \frac{\rho hr}{(1-a)^2} \int_{-\alpha}^{\alpha} [1-a\cos\phi]^3 d\phi$$

$$M_6 = \rho hr R^2 \int_{-\alpha}^{\alpha} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right]^2 [1-a\cos\phi]^3 d\phi$$

$$M_7 = \rho hr^3 \frac{4}{(1+a)^4} \int_{-\alpha}^{\alpha} \left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} + \right.$$

$$\left. \frac{a}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan(\phi/2)\right]\right]^2 [1-a\cos\phi]^3 d\phi$$

$$\simeq \rho hr^3 \int_{-\alpha}^{\alpha} [\sin\phi + a(\phi + \frac{1}{2}\sin 2\phi) + a^2\sin\phi(2+\cos^2\phi] \times$$

$$[1-a\cos\phi]^3 d\phi$$

$$M_8 = \rho hr^3 \frac{4}{(1+a)^2} \int_{-\alpha}^{\alpha} \frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} +$$

$$\left. \frac{1}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan(\phi/2)\right]\right]^2 [1-a\cos\phi]^3 d\phi$$

$$\simeq \rho hr^3 \int_{-\alpha}^{\alpha} [\phi + 2a\sin\phi + \frac{3}{2}a^2(\phi + \frac{1}{2}\sin 2\phi)]^2 [1-a\cos\phi]^3 d\phi$$

$$M_9 = \frac{\rho hr}{R[1-a]} \int_{-\alpha}^{\alpha} \left[\frac{1}{(1-a)} - \frac{1}{(1-a\cos\phi)}\right] (1-a\cos\phi)^3 d\phi$$

$$M_{10} = \rho h r^3 \frac{4}{(1+a)^2} \int_{-\alpha}^{\alpha} \left[\frac{(1+a)\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} + \right.$$

$$\left. \frac{a}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan(\phi/2)\right] \right] \times$$

$$\left[\frac{(1+a)a\tan(\phi/2)}{(1-a)[(1-a) + (1+a)\tan^2(\phi/2)]} + \right.$$

$$\left. \frac{1}{(1+a)} \left(\frac{1+a}{1-a}\right)^{3/2} \tan^{-1}\left[\left(\frac{1+a}{1-a}\right)^{1/2} \tan(\phi/2)\right] \right] [1-a\cos\phi]^3 d\phi$$

$$\simeq \rho h r^3 \int_{-\alpha}^{\alpha} [\sin\phi + a(\phi + \tfrac{1}{2}\sin 2\phi) + a^2 \sin\phi(\cos^2\phi + 2)] \times$$

$$[\phi + 2a\sin\phi + \tfrac{3}{2} a^2(\phi + \tfrac{1}{2}\sin 2\phi)][1-a\cos\phi]^3 d\phi$$

$$M_{11} = \rho h r \int_{-\alpha}^{\alpha} \cos^2\phi(1-a\cos\phi)d\phi$$

$$M_{12} = \rho h r \int_{-\alpha}^{\alpha} \sin^2\phi(1-a\cos\phi)d\phi$$

The following table contains comparisons of I_i and M_i for the Simpson's rule solutions and the binomial series expansion of the integrand solutions for $a = 0.2$, $h = 0.05"$, and $\phi_o = 10°$.

Table 3. Comparisons of Simpson's Rule Integrals and Binomial Series Expansion Integrals for a = 0.20, h = 0.05", and $\phi_o = 10^o$

	Simpson's Rule	Binomial Series Expansion
I_1	4.60	4.46
I_2	565.0	550.0
I_3	373.0	342.0
I_4	1410.0	1340.0
I_5	-43.8	-42.5
I_6	677.0	633.0
I_7	-0.00123	0.00122
I_8	-0.000634	0.000634
I_9	-0.0745	0.0736
I_{10}	-0.650	-0.644
I_{11}	0.476	0.466
I_{12}	0.792	0.782
I_{13}	1.59	1.59
I_{14}	1.49	1.47
I_{15}	3.07	3.06
I_{16}	0.471	0.460
I_{17}	0.768×10^{-4}	0.764×10^{-4}

Table 3. Comparisons of Simpson's Rule Integrals and Binomial Series Expansion Integrals for a = 0.20, h = 0.05", and $\phi_o = 10°$ (Continued)

	Simpson's Rule	Binomial Series Expansion
I_{18}	0.250×10^{-9}	0.241×10^{-9}
I_{19}	0.257×10^{-7}	0.254×10^{-7}
I_{20}	0.192×10^{-7}	0.176×10^{-7}
I_{21}	0.208×10^{-8}	0.203×10^{-8}
I_{22}	0.656×10^{-7}	0.636×10^{-7}
I_{23}	0.331×10^{-7}	0.310×10^{-7}
I_{24}	0.442×10^{-6}	0.439×10^{-6}
I_{25}	0.135×10^{-5}	0.130×10^{-5}
I_{26}	0.179×10^{-5}	0.174×10^{-5}
I_{27}	0.899×10^{-6}	0.817×10^{-6}
I_{28}	0.587×10^{-7}	0.506×10^{-7}
I_{29}	-0.530×10^{-6}	-0.546×10^{-6}
I_{30}	0.807×10^{-5}	0.794×10^{-5}
I_{31}	0.397×10^{-5}	0.377×10^{-5}
I_{32}	0.104×10^{-7}	0.102×10^{-7}
I_{33}	0.782×10^{-7}	0.780×10^{-7}
I_{34}	0.176×10^{-7}	0.153×10^{-7}
I_{35}	-0.413×10^{-7}	-0.467×10^{-7}
I_{36}	0.134×10^{-6}	0.127×10^{-6}
I_{37}	0.226×10^{-6}	0.222×10^{-6}

Table 3. Comparisons of Simpson's Rule Integrals and Binomial Series Expansion Integrals for a = 0.20, h = 0.05", and ϕ_o = 10° (Continued)

	Simpson's Rule	Binomial Series Expansion
I_{38}	0.123×10^{-5}	0.123×10^{-5}
I_{39}	0.209×10^{-7}	0.185×10^{-7}
I_{40}	-0.196×10^{-7}	-0.233×10^{-7}
I_{41}	0.587×10^{-7}	0.507×10^{-7}
I_{42}	-0.530×10^{-6}	-0.546×10^{-6}
I_{43}	0.809×10^{-5}	0.808×10^{-5}
I_{44}	0.253×10^{-6}	0.237×10^{-6}
M_1	0.000406	0.000406
M_2	0.000356	0.000353
M_3	0.000762	0.000759
M_4	-0.280×10^{-4}	-0.274×10^{-4}
M_5	-0.00123	0.00117
M_6	0.183	0.176
M_7	0.103	0.0912
M_8	0.444	0.416
M_9	0.0135	0.0129
M_{10}	0.200	0.182
M_{11}	0.000356	0.000353
M_{12}	0.000406	0.000406

APPENDIX C

INTERMEDIATE INTEGRALS

The approximate integrals of the integrand binomial series expansion solution are linear combinations of the following integrals.

$$A_1 = \int_{-\alpha}^{\alpha} \cos^3\phi \sin^4\phi d\phi = \frac{2}{7} \sin^5\alpha\left(\frac{2}{5} + \cos^2\alpha\right)$$

$$A_2 = \int_{-\alpha}^{\alpha} \cos^4\phi \sin^4\phi d\phi = \frac{1}{4} \cos^3\alpha \sin^5\alpha -$$

$$\frac{3}{8}\left[\frac{1}{3} \sin^3\alpha \cos^3\alpha - \frac{1}{8}(\alpha - \frac{1}{4} \sin 4\alpha)\right]$$

$$A_3 = \int_{-\alpha}^{\alpha} \cos^2\phi \sin^4\phi d\phi = -\frac{1}{3} \sin^3\alpha \cos^3\alpha + \frac{1}{8}(\alpha - \frac{1}{4} \sin 4\alpha)$$

$$A_4 = \int_{-\alpha}^{\alpha} \cos^3\phi \sin^2\phi d\phi = \frac{2}{5} \sin^3\alpha\left(\frac{2}{3} + \cos^2\alpha\right)$$

$$A_5 = \int_{-\alpha}^{\alpha} \sin^4\phi d\phi = -\frac{1}{2} \sin^3\alpha \cos\alpha + \frac{3}{4}(\alpha - \frac{1}{2} \sin^2\alpha)$$

$$A_6 = \int_{-\alpha}^{\alpha} \cos^3\phi d\phi = \frac{2}{3} \sin\alpha(2+\cos^2\alpha)$$

$$A_7 = \int_{-\alpha}^{\alpha} \cos^4\phi \sin^2\phi d\phi = \frac{1}{3} \cos^3\alpha \sin^3\alpha + \frac{1}{8}(\alpha - \frac{1}{4} \sin 4\alpha)$$

$$A_8 = \int_{-\alpha}^{\alpha} \cos^2\phi \sin^2\phi d\phi = \frac{1}{4}(\alpha - \frac{1}{4} \sin 4\alpha)$$

$$A_9 = \int_{-\alpha}^{\alpha} \cos^5\phi \sin^2\phi d\phi = \frac{2}{7} \sin^3\alpha \left\{\cos^4\alpha + \frac{4}{5}\cos^2\alpha + \frac{8}{15}\right\}$$

$$A_{10} = \int_{-\alpha}^{\alpha} \cos^5\phi d\phi = \frac{2}{5}\cos^4\alpha\sin\alpha + \frac{8}{15}\sin\alpha(\cos^2\alpha+2)$$

$$A_{11} = \int_{-\alpha}^{\alpha} \cos^4\phi d\phi = \frac{1}{2}\cos^3\alpha\sin\alpha + \frac{3}{4}(\alpha + \frac{1}{2}\sin2\alpha)$$

$$A_{12} = \int_{-\alpha}^{\alpha} \cos^6\phi d\phi = \frac{1}{3}\cos^5\alpha\sin\alpha + \frac{5}{12}\cos^3\alpha\sin\alpha + \frac{5}{8}(\alpha + \frac{1}{2}\sin2\alpha)$$

$$A_{13} = \int_{-\alpha}^{\alpha} \cos^2\phi d\phi = \alpha + \frac{1}{2}\sin2\alpha$$

$$A_{14} = \int_{\alpha}^{\alpha} \cos\phi d\phi = 2\sin\alpha$$

$$A_{15} = \int_{-\alpha}^{\alpha} \cos^6\phi\sin^2\phi d\phi = \frac{1}{4}\cos^5\alpha\sin^3\alpha + \frac{5}{12}\cos^3\alpha\sin^3\alpha +$$

$$\frac{5}{64}(\alpha - \frac{1}{4}\sin4\alpha)$$

$$B_1 = \int_{-\alpha}^{\alpha} \phi\sin^3\phi\cos^3\phi d\phi = \frac{\alpha}{6}\sin^4\alpha(1+2\cos^2\alpha) +$$

$$\frac{1}{6}\left[\frac{1}{3}\sin^3\alpha\cos^3\alpha - \frac{1}{8}(\alpha - \frac{1}{4}\sin4\alpha)\right] +$$

$$\frac{1}{24}[\sin^3\alpha\cos\alpha - \frac{3}{2}(\alpha - \frac{1}{2}\sin2\alpha)]$$

$$B_2 = \int_{-\alpha}^{\alpha} \phi\sin^3\phi d\phi = -\frac{2}{3}\alpha\cos\alpha(\sin^2\alpha+2) + \frac{2}{3}\sin\alpha(2 + \frac{1}{3}\sin^2\alpha)$$

$$B_3 = \int_{-\alpha}^{\alpha} \phi\sin^3\phi\cos^2\phi d\phi = \frac{2}{5}\,\alpha[\cos\alpha\sin^4\alpha - \frac{1}{3}\cos\alpha(\sin^2\alpha+2)] -$$

$$\frac{1}{5}\left[\frac{2}{5}\sin^5\alpha - \frac{2}{3}\,(2\sin\alpha + \frac{1}{3}\sin^3\alpha)\right]$$

$$B_4 = \int_{-\alpha}^{\alpha} \phi\sin^3\phi\cos^4\phi d\phi = -\frac{2}{7}\,\alpha\cos^5\alpha\left(\frac{2}{5} + \sin^2\alpha\right) +$$

$$\frac{1}{7}\left[\frac{2}{7}\sin^3\alpha\left(\cos^4\alpha + \frac{4}{5}\cos^2\alpha + \frac{8}{15}\right) + \frac{4}{25}\sin\alpha(\cos^4\alpha + \frac{4}{3}\,[2+\cos^2\alpha])\right]$$

$$B_5 = \int_{-\alpha}^{\alpha} \phi\sin\phi\cos\phi d\phi = \alpha\sin^2\alpha - \frac{1}{2}\,(\alpha - \frac{1}{2}\sin2\alpha)$$

$$B_6 = \int_{-\alpha}^{\alpha} \phi\sin\phi\cos^2\phi d\phi = -\frac{2}{3}\,\alpha\cos^3\alpha + \frac{2}{9}\sin\alpha(2+\cos^2\alpha)$$

$$B_7 = \int_{-\alpha}^{\alpha} \phi\sin\phi\cos^3\phi d\phi = -\frac{1}{2}\,\alpha\cos^4\alpha + \frac{1}{8}\,[\cos^3\alpha\sin\alpha + \frac{3}{2}\,(\alpha + \frac{1}{2}\sin2\alpha)]$$

$$B_8 = \int_{-\alpha}^{\alpha} \phi\sin\phi\cos^4\phi d\phi = -\frac{2}{5}\,\alpha\cos^5\alpha + \frac{2}{25}\sin\alpha[\cos^4\alpha + \frac{4}{3}\sin\alpha(2+\cos^2\alpha)]$$

$$B_9 = \int_{-\alpha}^{\alpha} \phi\sin\phi\cos^5\phi d\phi = -\frac{1}{3}\,\alpha\cos^6\alpha + \frac{1}{18}\cos^5\alpha\sin\alpha +$$

$$\frac{5}{72}\,[\cos^3\alpha\sin\alpha + \frac{3}{2}\,(\alpha + \frac{1}{2}\sin2\alpha)]$$

$$B_{10} = \int_{-\alpha}^{\alpha} \phi\sin^3\phi\cos\phi d\phi = \frac{1}{2}\,\alpha\sin^4\alpha + \frac{1}{8}\sin^3\alpha\cos\alpha - \frac{3}{16}\,(\alpha - \frac{1}{2}\sin2\alpha)$$

$$B_{11} = \int_{-\alpha}^{\alpha} \phi\sin\phi d\phi = 2(\sin\alpha - \alpha\cos\alpha)$$

$$C_1 = \int_{-\alpha}^{\alpha} \phi^2 \sin^2\phi \cos^2\phi \, d\phi = \frac{1}{4}\,\alpha^2\left(\alpha - \frac{1}{4}\sin 4\alpha\right) - \frac{1}{6}\,\alpha^3 +$$

$$\frac{1}{32}\left(\frac{1}{4}\sin 4\alpha - \alpha\cos 4\alpha\right)$$

$$C_2 = \int_{-\alpha}^{\alpha} \phi^2 \sin^2\phi \cos^3\phi \, d\phi = \frac{2}{5}\,\alpha^2 \sin^3\alpha\left(\frac{2}{3} + \cos^2\alpha\right) +$$

$$\frac{4}{25}\left[\alpha\cos^3\alpha\left(\frac{2}{3} + \sin^2\alpha\right) - \frac{1}{5}\sin^3\alpha\left(\frac{2}{3} + \cos^2\alpha\right) - \frac{2}{9}\sin\alpha(2+\cos^2\alpha)\right] +$$

$$\frac{8}{45}\left[\alpha\cos\alpha(2+\sin^2\alpha) - \sin\alpha\left(2 - \frac{1}{3}\sin^2\alpha\right)\right]$$

$$C_3 = \int_{-\alpha}^{\alpha} \phi^2 \sin^2\phi \cos^4\phi \, d\phi = \alpha^2\left[\frac{1}{3}\cos^3\alpha\sin^3\alpha + \frac{1}{8}\left(\alpha - \frac{1}{4}\sin 4\alpha\right)\right] -$$

$$\frac{1}{18}\left[\alpha\sin^4\alpha(1+2\cos^2\alpha) + \frac{1}{3}\sin^3\alpha\cos^3\alpha - \frac{1}{8}\left(\alpha - \frac{1}{4}\sin 4\alpha\right)\right] -$$

$$\frac{1}{72}\left[\sin^3\alpha\cos\alpha - \frac{3}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)\right] - \frac{1}{12}\,\alpha^3 + \frac{1}{64}\left[\frac{1}{4}\sin 4\alpha - \alpha\cos 4\alpha\right]$$

$$C_4 = \int_{-\alpha}^{\alpha} \phi^2 \cos\phi \, d\phi = 2\left[2\alpha\cos\alpha + (\alpha^2-2)\sin\alpha\right]$$

$$C_5 = \int_{-\alpha}^{\alpha} \phi^2 \cos\phi \, d\phi = \alpha^2\left(\alpha + \frac{1}{2}\sin 2\alpha\right) - \frac{2}{3}\,\alpha^3 - \alpha\sin^2\alpha + \frac{1}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$$

$$D_1 = \int_{-\alpha}^{\alpha} \sin^4\phi \, d\phi = -\frac{1}{2}\sin^3\alpha\cos\alpha + \frac{3}{4}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$$

$$D_2 = \int_{-\alpha}^{\alpha} \sin^4\phi \cos\phi \, d\phi = \frac{2}{5}\sin^5\alpha$$

$$D_3 = \int_{-\alpha}^{\alpha} \sin^2\phi\cos\phi d\phi = \frac{2}{3}\sin^3\alpha$$

$$D_4 = \int_{-\alpha}^{\alpha} \sin^2\phi d\phi = \alpha - \frac{1}{2}\sin 2\alpha$$

$$D_5 = \int_{-\alpha}^{\alpha} d\phi = 2\alpha$$

$$D_6 = \int_{-\alpha}^{\alpha} \phi^2 d\phi = \frac{2}{3}\alpha^3$$

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